Generalized Multivariable Alexander Invariants for Virtual Knots and Links

by

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Abstract

Virtual knot theory was introduced by Louis Kauffman in 1996 as a generalization of classical knot theory. This has motivated the development of invariants for virtual knots which generalize classical knot invariants. After providing an overview of knot theory, we study Alexander invariants (i.e., module, polynomial) and their generalization to virtual knots. Several papers ([**Saw**],[**SW1**],[**Ma1**],[**KR**],[**FJK**], [**BF**]) have previously addressed this topic to varying extents. The title of this paper has dual meaning: Based on the work of [**FJK**], we describe an intuitive approach to discovering a two variable polynomial invariant $\Delta_i(s,t)$ which we call the *generalized Alexander polynomial*—named such because setting $s = 1$ returns the classical Alexander polynomial (Section 6). Furthermore, after showing how the invariants discussed by other authors are specializations of this polynomial (Section 7), we introduce a formulation of the *generalized multivariable Alexander polynomial* $\Delta_i(s, t_1, \ldots, t_l)$ for virtual links in a manner which is not presently found in the literature (Section 8). Lastly, we use [**Pol**] to more efficiently address the subtleties of oriented Reidemeister moves involved in establishing the invariance of the aforementioned polynomials (Section 9).

KEY WORDS: multivariable Alexander polynomial, multivariable biquandle, virtual links, Alexander switch

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—Alex P. Miller

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CONTENTS

INTRODUCTION

Reading Guide

Part I[1] of this paper provides an elementary introduction to classical and virtual knot theory. This portion of the paper is intended for the lay reader with little to no background in mathematics. After describing the basic notions of knot theory in Sections 1 through 3, Section 4 provides a description of the current research project.

Part II is the major research component of this paper and is written for those familiar with knot theory. Congruous with an understanding of knot theory, the reader should be reasonably familiar with both algebraic (group, ring, and module theory) and topological (homeomorphism, fundamental group) notions. The motivated reader is encouraged to continue reading even if they are not familiar with every result stated in this paper. This paper is an appropriate read for the undergraduate mathematics student with an interest in the subject. Suggestions for background reading are provided where appropriate.

Section 5 describes the derivation of the classical Alexander polynomial via Fox's free calculus. Section 6 provides an intuitive description of how one might arrive at the definition of the generalized Alexander polynomial $\Delta_i(s,t)$. Section 7 provides a brief description of the relevant papers on the Alexander polynomial for virtual knots. In Section 8, we introduce and prove the invariance of the generalized multivariable Alexander polynomial $\Delta_i(s,t_1,\ldots,t_l)$, an invariant which extends the work of [**FJK**]. Section 9 addresses the subtleties associated with considering oriented Reidemeister moves and, lastly, Section 10 describes future possible avenues of research.

Executive Summary

Since its introduction more than 80 years ago, the Alexander polynomial has been an object of almost constant study in the classical theory of knots and links. Nonetheless, new realizations about the invariant are still being brought to light. Just in the past 15 years, several interesting and surprising results have been discovered between the Alexander polynomial and other invariants (i.e., the Jones polynomial, cf. [**BG**], [**V**], [**Chm1**]; and also Milnor's linking numbers, cf. [**Le**],

[1] In the digital PDF of this document, clickable internal links are **blue** and external links are **green**.

[**Tr**], [**MV**]). The generalization of classical knots to virtual knots has prompted the question of which results in the classical theory have analogues in the more general setting of virtual knot theory. For example, it is unknown whether results similar to those just mentioned regarding the Alexander polynomial hold for virtual knots.

This is due, in part, to the fact that the mere generalization of the Alexander polynomial is still under research. Nonetheless, some interesting results have already been discovered regarding Alexander invariants for virtual knots. Sawollek [**Saw**], for example, demonstrated that the extension of the Alexander polynomial via virtual link groups satisfies no linear skein relation like in the classical case (thus the normalized Alexander-Conway polynomial appears to have no analogue for virtual knots). There is, however, a normalized two-variable polynomial $Z(x, y)$ introduced by Sawollek in the same paper (based on an invariant in [**JKS**] defined for knots on thickened surfaces) which does satisfy a skein relation. Each of these invariants, it turns out, coincide with the virtual Alexander polynomial $\Delta_i(u_1, \ldots, u_d, v)$ introduced by Silver and Williams in [**SW1**] via "extended Alexander groups". If we let $\Delta_i(u, v)$ be the polynomial obtained by setting $u_1 = \cdots = u_d = u$, then Sawollek's Alexander polynomial is equal to $\Delta_i(u, 1)$. Similarly, Sawollek's $Z(x, y)$ is closely related to $\Delta_0(u, v)$. (See [**SW2**] for more on this; Silver and Williams actually apply Sawollek's normalization to obtain a normalized version of their Δ_0 in this same paper.)

Silver and Williams' polynomial $\Delta_0(u, v)$ is, in turn, equivalent to the generalized Alexander polynomial $G_K(s, t)$ of Kauffman and Radford [**KR**]. The polynomial $\Delta_i(s, t)$ of Fenn, Jordan-Santana, and Kauffman [**FJK**] is also the same as Silver and Williams' $\Delta_i(u, v)$ (as much is acknowledged therein). These two papers, however, define their invariants in terms of *biquandles* and *switches*. This construction is itself a generalization of Silver and Williams' extended Alexander group. However, there is a discrepancy in this generalization. Silver and Williams' formulation of Δ_i allows for the assignment of distinct variables to each component in the case of links, whereas [**FJK**]'s polynomial is only defined in two variables. There is no mention of a *multivariable* Alexander switch or biquandle for several-component virtual links.

After describing an intuitive approach to the theory of switches and biquandles (Section 6) and providing a more descriptive survey of the literature on virtual Alexander polynomials (Section 7), we address the absence of a multivariable theory of Alexander switches/biquandles (Section 8). We introduce what we call the *generalized multivariable Alexander polynomial* $\Delta_i(s, t_1, \ldots, t_l)$ for

virtual links. Though this polynomial coincides with Silver and Williams' virtual Alexander polynomial $\Delta_i(u_1, \ldots, u_d, v)$, the formulation of this invariant in the language of switches/biquandles (based on the work of [**FJK**]) is nowhere to be found in the literature.

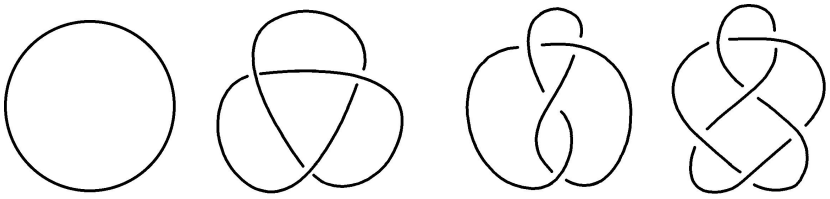
There is another subtlety involving the orientation of Reidemeister moves which is addressed in this paper. Though there are three classical Reidemeister moves, if we consider all the possible orientations of strands in these moves, there are actually 16 oriented Reidemeister moves. Thus, to prove a quantity is an invariant of oriented knots, we must take into account all 16 moves. This is briefly addressed in both [**FJK**] and Bartholomew and Fenn [**BF**], but only in passing and, perhaps, inadequately. In 2010, Michael Polyak [**Pol**] demonstrated that some sets of oriented Reidemeister moves generate all others, while other subsets do not. We apply Polyak's results to more efficiently address the invariance of Alexander invariants (Section 9).

PART I

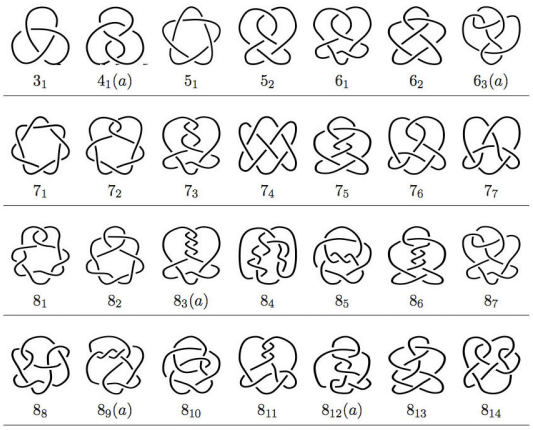
1 BACKGROUND INFORMATION

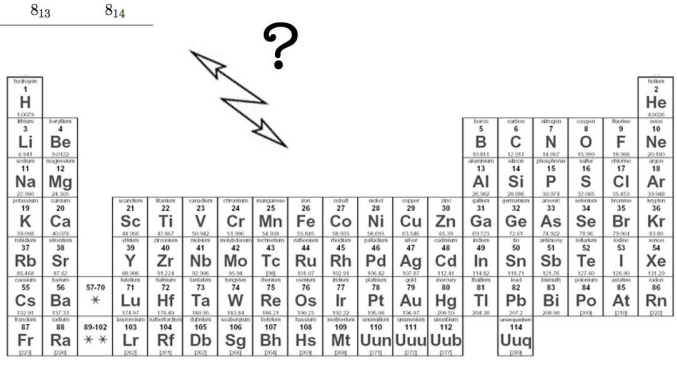
1.1 History of Knot Theory

What is now understood as the modern field of knot theory had its beginnings in the 1860s when the British physicist Lord Kelvin (1824-1907) hypothesized that atoms were knotted vortices in the ether of space. His theory was that each atom could be represented by a unique knot, where a knot in this case refers to a looped figure, like those represented below.



Scottish physicist Peter Tait (1831-1901) took interest in Kelvin's theory and attempted to tabulate all possible knots, believing that his table of knots would coincide with the table of elements in some way.





Are atomic elements tiny knots in the fabric of space?

However, the famous Michelson-Morley experiment of 1887 disproved the existence of ether, thereby

making Kelvin and Tait's physical interpretation of knots obsolete. Nonetheless the ideas and results of the physical theory of knots remained.

As the mathematical field of topology was developed in the early 20th century, several mathematicians reexamined Tait's results, which could now be formulated in precise mathematical language. (Topology is a field of mathematics involving the study of continuous transformations between objects.) And, as it turned out, the problems of knot theory—originally posed in the context of a flawed theory about the composition of matter—proved to be incredibly rich in mathematical allure and complexity. Throughout much of the 20th century, knot theory was investigated by mathematicians with varying degrees of intensity, with periods of elevated interest when new discoveries were made. It was in 1984 when interest in the field was significantly stimulated by Vaughn Jones' discovery of his polynomial knot invariant. The *Journal of Knot Theory and its Ramifications* was established in 1992 as the first academic journal dedicated purely to research in knot theory. Research in the subject has continued on a consistent basis over the past 30 years and our understanding of knot theory has substantially developed in both depth and breadth.[2]

1.2 Context and Applications

Within the the field of mathematics, knot theory sits at the intersection of low-dimensional topology, combinatorics, and algebra. Low-dimensional topology has been an area of particular interest in the past few decades; Vaughn Jones (mentioned above) as well as Edward Witten received Fields Medals[3] in 1990, largely for their work in this field. Their work (as well as that of others) has demonstrated unexpected relationships between knot theory and fields in physics such as statistical mechanics, quantum field theory, and string theory. Despite its abstract and seemingly inconsequential nature, knot theory has found serious applications in fields outside of mathematics. In addition to the relationship with physics mentioned above, biologists have used some tools from knot theory to study the effect of topoisomerases on DNA molecules. Knotting phenomena has also been found in the chemistry of molecule synthesization (see [**Ad**] for details about these applications). Additionally, several developments in recent years have shown that knot theory may have the potential to be an integral part of the theory of quantum computing [**Col**].

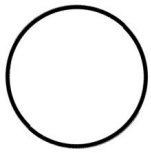
[2]For more on the history of knot theory, the reader is referred to [**Sil**].

[3]The Fields Medal is among the most prestigious awards given in the field of mathematics. It has been called the "Nobel Prize" of mathematics, but is awarded every four years and only to mathematicians under the age of 40.

2 ELEMENTARY INTRODUCTION TO KNOT THEORY

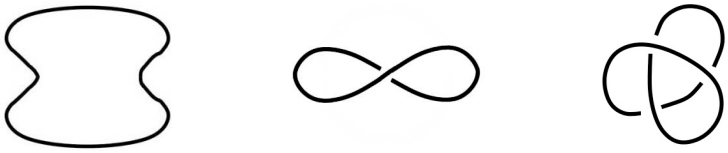
2.1 Knots and Links

A mathematical "knot" is more or less defined as any configuration of a string in three-dimensional space connected at its ends. The simplest knot is just a closed loop with no twists, called the "unknot".



The unknot

Two knots are said to be equivalent if one can be contorted and deformed to look like the other one without cutting the string or passing it through itself; two knots are distinct if no such deformation exists. Here, the notion of contorting and deforming knots is best understood by imagining that our string is infinitely stretchable, malleable, and one-dimensional (i.e., a line with infinitesimal thickness). The following knots are equivalent to the unknot because they can be untwisted and deformed to look like the circle above (and are thus called "trivial").



Trivial knots[4]

The main goal of knot theory is to classify and distinguish all knots up to equivalence. In other words, the most important question in knot theory is whether or not two knots are the same. This task is not as easy as it may seem. For example, exactly one of the knots below can actually be completely untangled to look like the unknot.

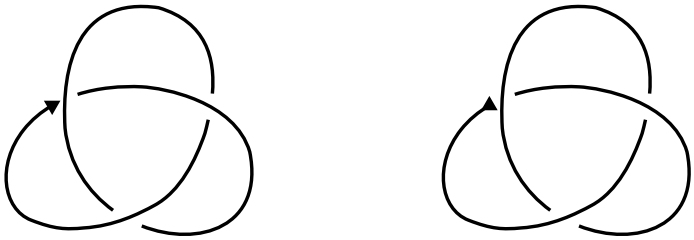
Which of these knots is trivial?

[4]These diagrams and several other in this section are adapted from [Chm2].

A natural generalization of the concept of a knot is that of a "link", i.e., two or more strands (called components) of (possibly knotted string) wrapped around each other. A link is similarly called trivial if it is equivalent to a set of unknotted loops (where equivalence is defined in the same way as for knots). From this point forward, we may use the word "knot" to refer to either a knot or a link.

Links of two and three components

Oftentimes, we like to think of knots as being *oriented*. An oriented knot is simply a knot like those already illustrated, but whose string in endowed with a directionality. This orientation is usually denoted by a simple arrow. Note that any un-oriented knot can be given one of two possible orientations.

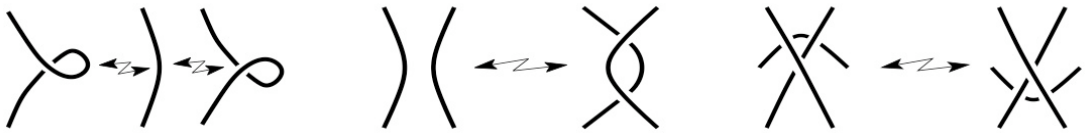


Two possible orientations of a knot

2.2 Knot Diagrams

In reality, knot theorists don't use literal pieces of string to model their ideas. Though the *idea* of a knot is three-dimensional, knot theorists actually use two-dimensional representations of knots like the ones already presented. These are called knot diagrams.

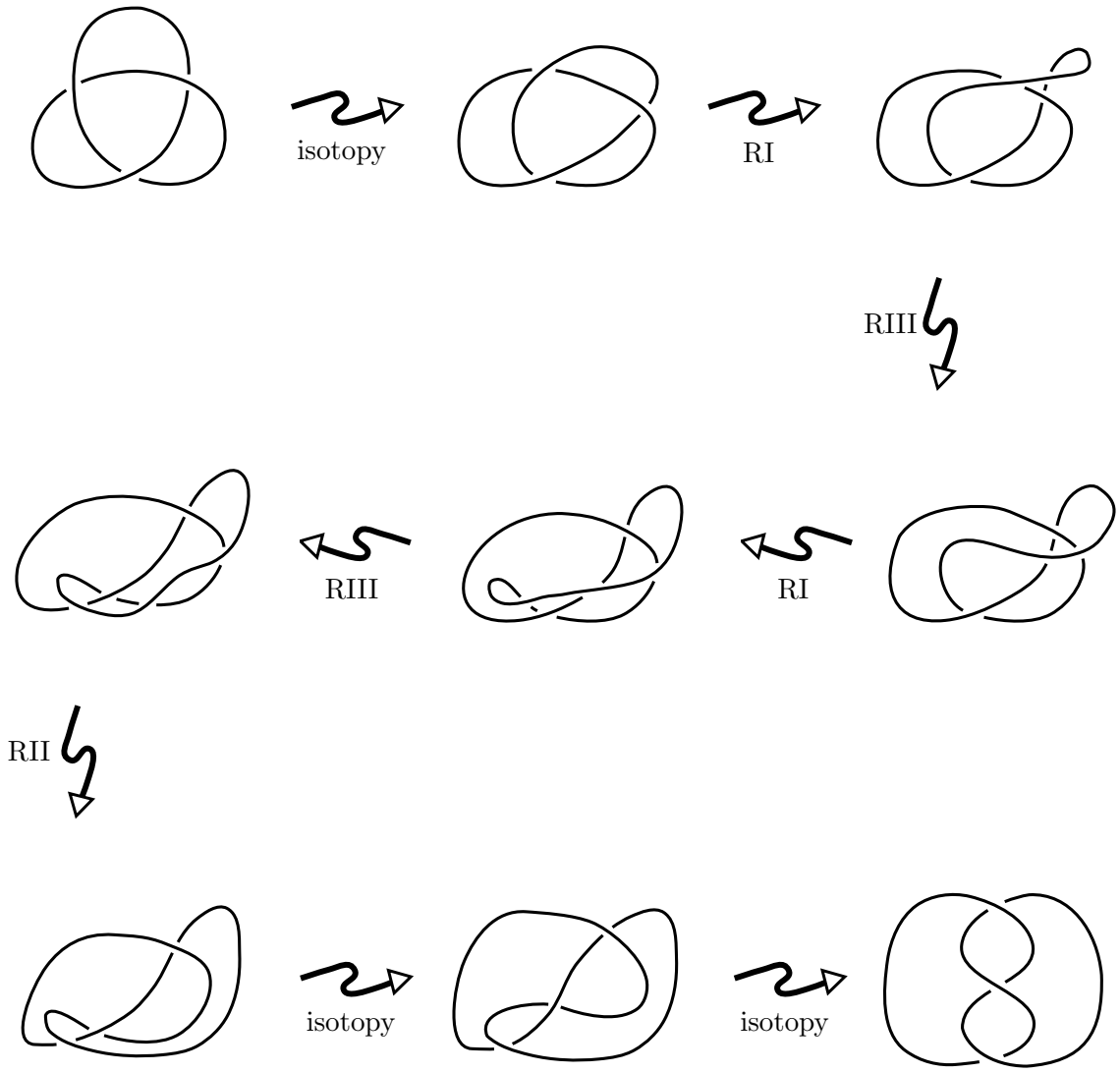
There is naturally some ambiguity with this transition from three dimensions to two. For example, in three-dimensions, I said knots are equivalent if one can be transformed to look like the other without cutting the string or passing it through itself. This same definition of equivalence cannot be directly applied to knot diagrams. Fortunately, there is a way to define equivalence of knot diagrams which coincides precisely with the intuitive idea of knot equivalence in three-dimensions. This is done using what are called the "Reidemeister moves" (represented below).



Type I **Type II** **Type III**

The images of the Reidemeister moves are meant to represent small regions of a knot diagram where the indicated transformation is being made. In 1926, the mathematician Kurt Reidemeister proved that given any two equivalent knots, their corresponding diagrams can be transformed to look like each other through a sequence of the three Reidemeister moves, applied to small portions of each diagram [**Reid**]. We will often refer to the Reidemeister moves as RI, RII, and RIII, respectively.

As a simple illustration of this concept, try to follow the next example step by step. Note that, in addition to the Reidemeister moves, we also allow transformations of the diagram which are analogous to stretching/shrinking/moving the string. The technical word for these kind of transformations is *isotopy*. Each diagram in the following sequence is equivalent to the succeeding one by the type of move indicated.



Because we've found a sequence of Reidemeister moves taking the first diagram to the last one, we say that they are equivalent diagrams—in the sense that they represent the same knot.

This result now gives us a well-defined notion of diagram equivalence, which agrees with our intuitive concept of knot equivalence in three dimensions. Specifically, we can say that two knot diagrams are equivalent if and only if there exists a sequence of Reidemeister moves transforming one to the other. (From now on, the term "knot" may be used synonymously with "knot diagram".)

It turns out that defining knots via diagrams and Reidemeister moves is more than just a convenient convention that allows us to draw them. Indeed, this approach has incredibly powerful implications for studying knots. We will see the power of the Reidemeister moves after we introduce

the notion of an invariant.

2.3 Invariants

The main tools that knot theorists use to distinguish knots are called *invariants*. An invariant is a way to code information about a knot that does not change when the knot's diagram changes. The general idea of an invariant is actually a very intuitive one. Consider the following trivial, but illustrative example.

Suppose you have a box of several hundred ladybugs. Take a ladybug out of the box and count the number of dots on its back; let's suppose it has 8 dots. Now put the ladybug back in the box. Choose another ladybug at random from the box. If this second ladybug does not have 8 spots, then we can conclude that it is not the same one as the first ladybug. In this way, we are using an invariant property about ladybugs (i.e., one that is not subject to change—the number of spots on their back) to distinguish them. We could say that the number of spots on a lady bug is an invariant of the ladybugs in our box.

Notice that our invariant does have some limitations, however. For example, suppose we choose a third ladybug at random and that it has 8 spots. Just because it has 8 spots is not enough information to conclude that it is the same bug as in our first draw. If we wanted to figure out whether or not this ladybug is the same as the first one, we would have to use a more specific invariant. One example might be the ladybugs' DNA. DNA is of course unique for each ladybug, however it is also much more difficult to analyze than the number of dots on a ladybug's back.

This example demonstrates two important properties about invariants in general. One is their specificity (i.e., their power to distinguish objects) and the other is their calculability. Just as in the ladybug example, these two properties usually inversely correlated—that is to say that the more specific or powerful our invariant is, the more difficult it is to calculate (and visa versa).

2.4 Invariants in Knot Theory

Invariants in knot theory (and in general) have very much in common with our ladybug example. There are many different properties that we can associate to each knot (we will give some examples in a moment). Some of these properties are easy to compute, while others are more difficult. Likewise, some of them can distinguish many knots, while others are less powerful. However, coming

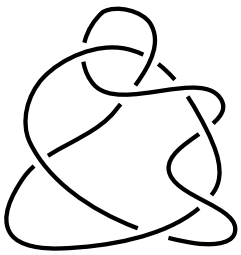
up with intrinsic properties about knots is considerably more difficult than finding invariants of ladybugs.

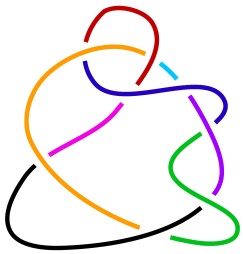
This is, indeed, a very hard problem and one which would be much more difficult without the aid of the Reidemeister moves. Recall that saying two knot diagrams are equivalent is the same as saying that one can be transformed to the other through a sequence of Reidemeister moves. Thus, a good definition of what a knot invariant is would be any property which is not subject to change when we change our knot diagram to an equivalent one, i.e., modify the diagram by any of the Reidemeister moves. The preceding sentence has an incredibly useful implication. That is, *in order to show that some property of a knot is invariant, we only need to show that this property does not change when we apply the Reidemeister moves to its knot diagram.*

To make this discussion a little more concrete, we will introduce one of the simplest knot invariants.

Tricoloarbility

In order to describe our first knot invariant, we will briefly introduce some more terminology. Given any knot diagram, we define its *arcs* as the portions of the diagram which are not intersected by any other part of the knot; an example demonstrates this notion most effectively. The following knot diagram has eight arcs, each of which is given a different color in the figure on the right.





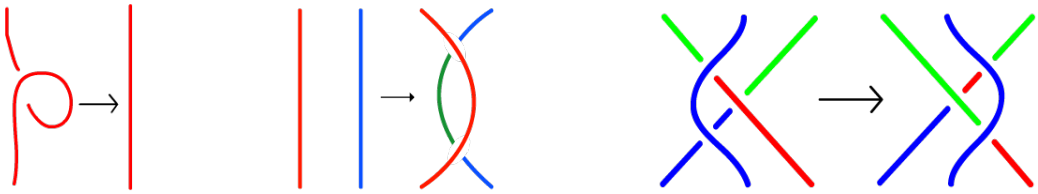
Our knot invariant can now be described in the following manner.

Definition[5]: A knot is said to be *tricolorable* if each arc of its knot diagram can be assigned one of three colors, satisfying the following conditions:

[5]This definition and the images in this section are taken from [**Wiki**]

- At least two colors must be used on the diagram
- At every point of intersection between arcs (or crossing), the three strands involved are either all the same color or all different colors

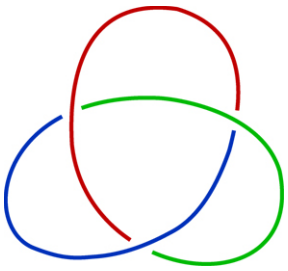
Recall that in order to prove some property is an invariant, we must prove that it does not change when we apply the Reidemeister moves to a knot diagram. Note that in each of the moves below, if a knot is tricolorable before applying the move, it will also be tricolorable after applying the move.



RI RII RIII

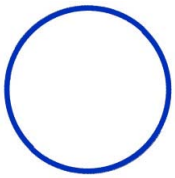
Thus, we have just shown that tricolorability is indeed a knot invariant. Think of tricolorability as a sort of knot thumbprint. If one diagram of a knot is tricolorable, then no matter how you twist and contort your knot, the result will still always be tricolorable. Similarly, if one diagram of a knot is not tricolorable, then neither will any other diagram.

If one knot is tricolorable and another is not, then we can conclude that it would be impossible to transform one knot into the other. This can be somewhat difficult to understand without trying some examples. As one simple example, consider the trefoil knot:

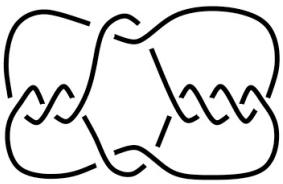


Note that we have colored the arcs of this knot according to the rules above, so we can say that trefoil is tricolorable. A good way to convince yourself that this is indeed an intrinsic invariant of the trefoil knot is to color each of the diagrams we considered on page 13 and make sure that each one is still tricolorable.

The unknot, just a closed loop, cannot be colored according to the rules specified in the definition. This is easy to see in its simplest form, where only one color can be used since there is only one arc.



However, even more complicated diagrams of the unknot still fail to be tricolorable. The following diagram (which we encountered before) is one such diagram of the unknot.



It is a good exercise to try to color the diagram with only three colors; you will find that, just like the simpler representation above, this diagram of the unknot is not tricolorable. Since the trefoil knot is tricolorable and the unknot is not tricolorable, we can definitively conclude that the these are distinct knots. This is the first time in this paper that we have actually established that any knot is non-trivial.

Polynomial Invariants

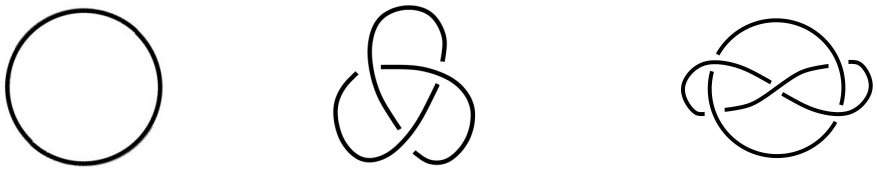
One of the most interesting and powerful class of knot invariants is that of polynomial invariants. A polynomial in this context mostly coincides with the concept introduced in many high school algebra classes.

$$t^4 - 4t^3 + 3t^2 + t - 1$$

The only difference between the polynomials referred to here is that the variable (t in the example above) is not necessarily meant to represent any number, but rather it functions more as a place-holder. Even though they have no numerical meaning, polynomials are still very useful because we can still perform operations on them (add, subtract, multiply, divide), as well as have a well-defined concept of equality between polynomials.

The basic concept behind polynomial invariants is this: given any knot, we can perform an algorithm on the knot in order to compute a polynomial. The resulting polynomial is then associated with that specific knot. If we can somehow prove that the polynomial will not change when the knot diagram changes, then we will have shown that the polynomial is indeed an invariant. Again, the Reidemeister moves make proving invariance very simple: we need only to prove that the invariant does not change under the three Reidemeister moves.

There are several polynomial invariants (sometimes referred to collectively as called knot polynomials), such as the Alexander polynomial, the Jones polynomial, and the HOMFLY polynomial; the Alexander polynomial is of particular interest in this paper. Invented by J.W. Alexander in 1928, the Alexander polynomial was the first polynomial knot invariant. Its original definition is quite complicated and involves the good deal of algebraic topology. Instead of going into the details here, let us just consider some examples. The Alexander polynomials of the following knots and links are listed below.



1 $t - 1 + t^{-1}$ $(t-1)(u-1)$

Alexander polynomials

To better understand what these polynomials mean, think back to the ladybug example. In this context, the value of the polynomial is like the number of dots on a ladybug's back. If two ladybugs have a different number of dots, then they are different ladybugs. In a similar fashion, if two knots have different Alexander polynomials, then they are different knots. We will describe the Alexander polynomial more explicitly in Section 5.2.

3 VIRTUAL KNOT THEORY

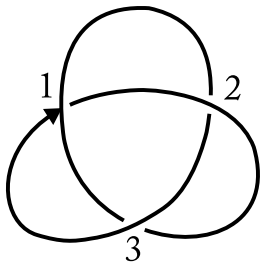
3.1 Gauss Codes

In 1996, Louis Kauffman (University of Illinois at Chicago) discovered a generalization of knot theory called "virtual" knot theory [**Kau**]. The motivation for virtual knot theory is best understood by examining what are called Gauss codes. As an alternative to simply drawing a knot, Gauss codes are a way of describing knots using a sequence of numbers. Gauss codes are useful because, though drawing knot diagrams are good visual representations for humans, diagrams are not good representations for computers. In order to perform calculations on knots with computers, we need a way of coding knots that can be put into a computer program—this is where Gauss codes come in.

In order to represent a knot by its Gauss code, give the knot an orientation and pick an arbitrary point on the knot. Trace the knot in the direction of the orientation and label each crossing point sequentially (1, 2, 3, ...) until each crossing has been labeled once. Once each crossing is numbered, retrace the knot, and as you come across each number, write the number in a list along with whether the strand of the knot you are tracing is the over-strand or the under-strand in the cross by writing a "O" or an "U" before the number.

overstrand **understrand**

Continue this process until each number is listed twice in your sequence. The resulting sequence is called the Gauss code of the knot. An example is demonstrated below, where both the starting point and orientation are designated by the single arrow.

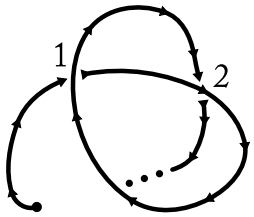


Gauss code: U1 O2 U3 O1 U2 O3

3.2 Virtualization

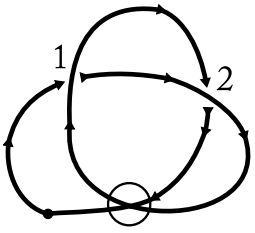
While every knot can be represented by a Gauss code, not every Gauss code can be drawn as a knot. For example, if we were to try to draw the knot of the following Gauss code, we would see that it is impossible to draw the knot without adding a new crossing.

Gauss code: U1 O2 O1 U2



The ends cannot be connected without adding a new crossing.

The problem with this is that Gauss codes are meant to include information about *every* crossing point on our diagram. Hence, if we were to connect the two ends of the above knot, we would necessarily add a crossing, thereby changing the Gauss code. Kauffman's solution to this problem is to connect the two ends of the knot, but indicating that the newly formed crossing is not part of its Gauss code. We do this by putting a circle around the new crossing and calling such a crossing "virtual".

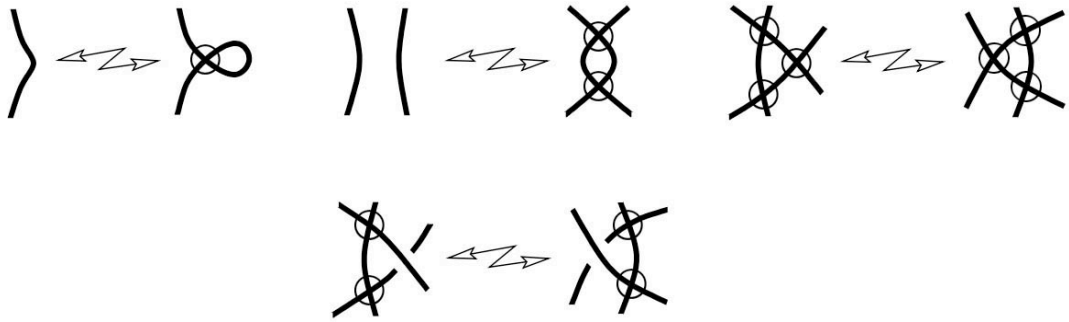


The same knot as above, but with its ends connected.

With this new convention, we can take *any* Gauss code and associate to it a virtual knot. Among

others, this is one possible motivation for virtual knot theory (see [**Kau**] for more).

The idea behind virtual crossings is that they are not really there. In order to compute the Gauss code of an arbitrary virtual knot, simply ignore the virtual crossing. The introduction of virtual crossings requires us to rethink what it means for two knots to be equivalent. This motivates the introduction of generalized virtual Reidemeister moves. The set of moves which we will allow on virtual know diagrams is illustrated below.



Virtual and Semi-virtual Reidemeister moves

The three moves involving only virtual crossings are simply called virtual Reidemeister moves; the fourth move in the set above is called *semi-virtual* because it involves both virtual and classical crossings.

An important result about virtual knot theory was proved by Polyak, Goussarov, and Viro [**GPV**]. They showed that any two classical knots (i.e., knots without virtual crossings) which are equivalent under these new generalized Reidemeister moves are also equivalent under the original Reidemeister moves. Hence, any results in virtual knot theory can be also be applied to classical knot theory. Thus, any study of virtual knots applies directly to classical knots.

4 DESCRIPTION OF RESEARCH

Having summarized the important concepts of knot theory and virtual knot theory, we can now more adequately describe the goals of this research paper. Just like for classical knots, invariants are the main object of study in virtual knot theory. Because we think of virtual crossings as not actually being there, one can often define virtual knot invariants in the same way as for classical knots by simply ignoring the presence of virtual crossings. This approach can be taken to define the Alexander polynomial on virtual knots. However, this invariant is not as strong as strong of an invariant for virtual knots as it is for classical knots (i.e., it does not detect the non-triviality of some simple virtual knots).

This has motivated the development of stronger invariants, one of which is the *generalized Alexander polynomial*. This generalization of the Alexander polynomial was actually independently encountered by several different authors ([**Saw**], [**SW1**], [**KR**], [**FJK**]). However, the routes by which these authors define their generalizations differ. In the present paper, we fill a gap between the work of [**SW1**] and [**FJK**] which exists due to their differing approaches. Specifically, we use the methods of [**FJK**] to define the multivariable version of the generalized Alexander polynomial. This invariant is equivalent to that of [**SW1**], but the hope is that defining this polynomial in multiple ways will provide insight into its properties which may not be obvious from existing approaches.

PART II

5 CLASSICAL KNOT INVARIANTS

A knot k is a simple closed curve in $\mathbb{R}^3$ endowed with an orientation. Let k_1 and k_2 be knots in $\mathbb{R}^3$. We say that k_1 and k_2 are *equivalent* (or *ambient isotopic*) if there exists an onto homeomorphism $f : \mathbb{R}^3 \to \mathbb{R}^3$ such that $f(k_1) = k_2$ which also preserves the orientation of $\mathbb{R}^3$. This definition of equivalence corresponds precisely with the intuitive notion derived by thinking of knots as loops of strings and saying that two knots are equivalent if one can be made to look like the other without cutting the string or passing it through itself.

5.1 The Knot Group

For any knot $k \subset \mathbb{R}^3$, we can consider the 3-manifold $\mathbb{R}^3 \setminus k$. We define the *knot group* of k to be $\pi_1(\mathbb{R}^3 \setminus k)$, the fundamental group of the complement of the knot.

Proposition 5.1: The knot group is a knot invariant (i.e., two knots k_1 and k_2 are equivalent only if they have isomorphic knot groups).

Proof: Assuming k_1 and k_2 are equivalent, there exists a homeomorphism $f : \mathbb{R}^3 \to \mathbb{R}^3$ such that $f(k_1) = k_2$. Hence, $f|_{\mathbb{R}^3 \setminus k_1} : \mathbb{R}^3 \setminus k_1 \to \mathbb{R}^3 \setminus k_2$ is a homeomorphism between the complements of k_1 and k_2. It is well known that homeomorphisms induce natural isomorphisms between fundamental groups. Thus, there exists an isomorphism $f_* : \pi_1(\mathbb{R}^3 \setminus k_1) \to \pi_1(\mathbb{R}^3 \setminus k_2)$, from which we see the knot groups must be isomorphic.

□

The knot group is a powerful invariant from which many other invariants (such as the Alexander polynomial) can be derived. It was at the beginning of the twentieth century that Wilhelm Wirtinger gave the first method for computing the knot group of an arbitrary knot by presentation (this method was first published in 1925) [**OR**]. Unfortunately, determining if two knot groups are isomorphic reduces to the "presentation problem" or "group isomorphism problem" (i.e., the fact that there is no algorithm to determine if two group presentations present isomorphic groups).

We will describe Wirtinger's original method for computing the knot group, which remains a very useful tool in knot theory. We will only describe the method for computing Wirtinger's presentation from a knot diagram, and not prove that the resulting group presentation is indeed

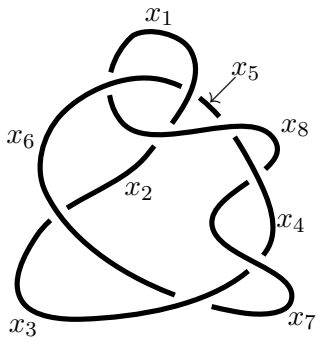
isomorphic to the knot group. A more detailed discussion on knot group presentations including such a proof can be found in [**CF**].

The Wirtinger Presentation of the Knot Group

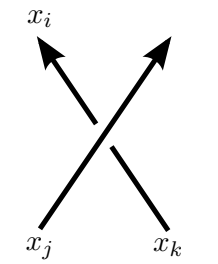
Given the diagram of a knot, we say a crossing is *positive* or *negative* depending on which strand is the overstrand; the rule is depicted below.

positive crossing negative crossing

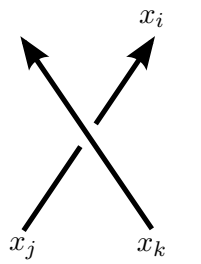
Given a knot diagram with n arcs, in order to present its knot group we label the arcs $x_1, \ldots, x_n$. After a labeling has been chosen, the crossing at which the i-th arc begins is referred to as the i-th crossing of the diagram. Consider the labeling of the following knot (whose orientation is implied by the numbering).



Observe that at any crossing of a knot diagram, exactly three arcs are involved. The Wirtinger presentation of a knot is given by n generators $x_1, \ldots, x_n$ (corresponding to the arcs of the knot) and n relations $r_1, \ldots, r_n$ (corresponding to its crossings). Each relation r_i is determined by considering the i-th crossing of the knot; if the arcs involved at this crossing are labeled x_i, x_j, and x_k, then the relation is given below, depending on whether the crossing is positive (left figure) or negative (right figure).



$r_i = x_i^{-1} x_j x_k x_j^{-1}$



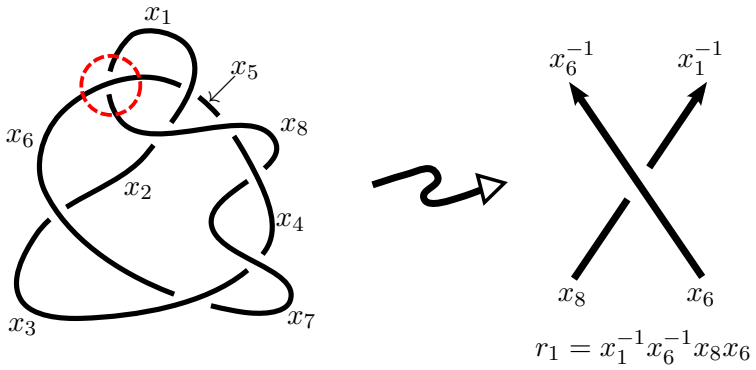
$r_i = x_i^{-1} x_k^{-1} x_j x_k$

A useful mnemonic for these rules is to travel in a small counterclockwise circle around the crossing, starting at the exiting understrand (x_i for the i-th crossing). Write your relator as a product of the variables associated to each strand you cross according to the following rule:

- If the strand is pointing *inward*, write the variable associated to that strand.
- If the strand is pointing *outward*, write the *inverse* of the variable associated to that strand.

One can also think of these relations being given by conjugation. The relation at a positive crossing can be written $x_i = x_j x_k x_j^{-1}$ and the relation at a negative crossing would be $x_i = x_k^{-1} x_j x_k$. As can be seen, this understanding has the benefit of a certain kind of symmetry.

The first relation in our continuing example is shown below.



$r_1 = x_1^{-1} x_6^{-1} x_8 x_6$

Completing the algorithm on this example yields a total of eight relations and the Wirtinger presentation of the knot group of this knot is given by

$$\left\langle \; x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 \;\middle|\; \begin{array}{ll} x_1^{-1} x_6^{-1} x_8 x_6 & x_2^{-1} x_8 x_1 x_8^{-1} \\ x_3^{-1} x_6^{-1} x_2 x_6 & x_4^{-1} x_7^{-1} x_3 x_7 \\ x_5^{-1} x_8^{-1} x_4 x_8 & x_6^{-1} x_1^{-1} x_5 x_1 \\ x_7^{-1} x_3^{-1} x_6 x_3 & x_8^{-1} x_4^{-1} x_7 x_4 \end{array} \right\rangle$$

5.2 The Alexander Polynomial

History

The Alexander polynomial has a rich and complex history which can be difficult to sort out at times. Over the years, several different definitions of the Alexander polynomial have been developed, using what seem like very different methods from topology, combinatorics, and algebra. There are at least three different methods to define the Alexander polynomial, and even more ways to compute it; it is somewhat remarkable that all these definitions are equivalent. Unfortunately, in the literature, any one of these definitions is often used to define *the* Alexander polynomial, without any reference to the other possible formulations. This can easily leave readers confused as to what exactly the Alexander polynomial is. An understanding of the history of the Alexander polynomial should clear up any of this confusion, as well as further contextualize the present research.

As a mathematician at Princeton University, J.W. Alexander introduced the first polynomial knot invariant—which he simply denoted by Δ—in 1928. His original definition of Δ was in terms of the first homology of the infinite abelian cover of the knot complement. In his paper, *Topological Invaraints of Knots and Links* [**Ale**], Alexander gives a combinatorial way to practically compute Δ for any knot as the normalized determinant of an incidence matrix constructed from a knot diagram. In 1934, Herbert Seifert showed how Δ could be computed via Seifert surfaces and Seifert matrices [**Sei**]. In the 1950's, Ralph H. Fox published a series of papers on the "free differential calculus" (or simply the "Fox calculus") of free groups [**Fox**]. Fox proved that Δ can be computed by considering the elementary ideals of a particular module (we will describe Fox's calculation in more detail later). In the 1960's, the prominent mathematician John H. Conway made an important contribution to our understanding of the Alexander polynomial [**Con**]. Conway demonstrated how Δ could be calculated via a *skein relation*; so important was Conway's contribution that the Alexander polynomial is sometimes called the Alexander-Conway polynomial. The discovery of the Jones polynomial in 1984 [**Jon**] stole the spotlight from the Alexander polynomial as the only knot polynomial, but the subsequent surge of interest knot theory only served to further our understanding of the Alexander polynomial and its relationship with other knot invariants. Another knot polynomial, the HOMFLY polynomial, was discovered soon after. It was found that both the Alexander and Jones polynomials could be derived from the HOMFLY polynomial [**LM**].

In 1996, the proof of the Melvin-Morton conjecture demonstrated a very deep relationship between the Alexander polynomial and the Jones polynomial ([**BG**], [**Lin**]).

In the present context, we will be interested in how this history continues and carries over to the world of virtual knots. However, before examining the more recent work in this area, it is important to have a grasp of the classical theory of the Alexander polynomial. We turn to this subject now, providing a development of Ralph H. Fox's theory of the Alexander polynomial. The reader is referred to the texts [**CF**] and [**Lck**] for more detailed expositions on the subject.

The Alexander Polynomial from the Fox Calculus on the Knot Group

Fox's derivation of the Alexander polynomial from the knot group is given here both as an instructive device for the interested reader, as well to better understand the classical context to which we refer when we consider the development of Alexander invariants for virtual knots.

Preliminary Definitions: Let $F = \langle x_1, \ldots, x_n \rangle$ be the free group of rank n and let $\mathbb{Z}F$ be the *integral group ring* of F. Typical elements in $\mathbb{Z}F$ are finite formal linear combinations of the form $\sum n_i w_i$, where $n_i \in \mathbb{Z}$ and $w_i \in F$. The ring structure is given by the two operations:

$$\begin{array}{ll} \text{Addition:} & \sum n_i g_i + \sum m_i g_i := \sum (n_i + m_i) g_i \\ \text{Multiplication:} & (\sum n_i g_i)(\sum m_j h_j) := \sum_{i,j} (n_i m_j) g_i h_j \end{array}$$

We define the *(partial) Fox derivative with respect to* x_j (or the j-th Fox derivative) as the map $\dfrac{\partial}{\partial x_j} : F \longrightarrow \mathbb{Z}F$ such that

- $\dfrac{\partial x_i}{\partial x_j} = \delta_{ij}$ (the Kronecker delta function)
- $\dfrac{\partial 1}{\partial x_j} = 0$
- $\dfrac{\partial gh}{\partial x_j} = \dfrac{\partial g}{\partial x_j} + g \dfrac{\partial h}{\partial x_j}$ where $g, h \in F$ (i.e., a modified product rule)

Note that this map (and, indeed, any homomorphism of F) can be extended to $\mathbb{Z}F$ by requiring

that it behave linearly with respect to addition.

$$\frac{\partial}{\partial x_j}(g+h) = \frac{\partial g}{\partial x_j} + \frac{\partial h}{\partial x_j} \quad \text{for} \quad g+h \in \mathbb{Z}F$$

Let G be an arbitrary group given by a presentation $\langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle$. Construct $\mathbb{Z}G$ from G in the same way we constructed $\mathbb{Z}F$ from F. Let $\varphi : \mathbb{Z}F \to \mathbb{Z}G$ be the relation homomorphism, sending relators r_i to $0 \in \mathbb{Z}G$.[6] Lastly, let $\alpha : \mathbb{Z}G \to \mathbb{Z}G_{ab}$ be the canonical abelianization homomorphism, where $\mathbb{Z}G_{ab} = \mathbb{Z}G/[\mathbb{Z}G, \mathbb{Z}G]$.

We now consider the chain of maps, $\alpha \circ \varphi \circ \dfrac{\partial}{\partial x_j}$ (see diagram below).

$$\mathbb{Z}F \xrightarrow{\frac{\partial}{\partial x_j}} \mathbb{Z}F \xrightarrow{\varphi} \mathbb{Z}G \xrightarrow{\alpha} \mathbb{Z}G_{ab}$$

Construct a matrix by considering how this chain acts of the relators of G, considered as elements in $\mathbb{Z}F$. Specifically, let $A = (a_{ij})$, where $a_{ij} \in \mathbb{Z}G_{ab}$ are given by

$$a_{ij} = \alpha\Big(\varphi\Big(\frac{\partial r_i}{\partial x_j}\Big)\Big)$$

We call this matrix the *Alexander matrix* of our group; it has m rows corresponding to the generators of G, and n columns corresponding to the relators.

If the group G is a knot group, a particularly nice result allows us to specify the ring in which the entries of the Alexander matrix of G lie. Specifically, for any knot group G, its abelianization G_{ab} is infinite cyclic (i.e., isomorphic to $\mathbb{Z}$) (see [**BZ**], p. 34). This means that the abelianizing step in the computation of A has the effect of sending all the generators $x_1, \ldots, x_n$ to a single generator, which we will denote by t. From this, we see that the Alexander matrix of k will have entires from the ring $\mathbb{Z}[t^{\pm 1}]$.[7]

[6] In terms of group presentation, we think of the relations r_i as words constructed from the generators $x_1, \ldots, x_n$ and then set to equal the unity $1 \in G$. However, when we think of r_i in terms of the module $\mathbb{Z}G$, it is easiest to think of them being sent to zero in $\mathbb{Z}G$. Thus, technically, the relators in $\mathbb{Z}G$ would be equal to $r_i - 1 = 0$ (or an equivalent formulation). However, we avoid making this distinction each time and trust that the reader will take note of the context in which the reference lies, thinking of relators being sent to 1 or zero where appropriate. As an example, a relator of G might look like $x_1 x_2 x_3^{-1} = 1$; when considered as a relator in $\mathbb{Z}G$ we would then think of it as $x_1 x_2 x_3^{-1} - 1 = 0$ or $x_1 x_2 - x_3 = 0$.

[7] Here, $\mathbb{Z}[t^{\pm 1}]$ is meant to indicate the integral group ring of an infinite cyclic group $< t >$. We prefer the notation

Note that this matrix A can be used to construct a module. We can consider each row as a relation between the generators $\mathbf{x} = (x_1 \cdots x_n)$ of $\mathbb{Z}F$. The matrix equation

$$A\mathbf{x} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ & \vdots & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \mathbf{0}$$

gives rise to m relators on the free module $\mathbb{Z}F$, which we could then present as

$$\left\langle x_1, \ldots, x_n \;\middle|\; \begin{array}{c} a_{11}x_1 + \cdots + a_{1n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = 0 \end{array} \right\rangle$$

If the group G is the knot group of a knot k, then this module is actually isomorphic to the *Alexander module* of k. If X_∞ is the infinite cyclic cover of the knot complement $\mathbb{R}^3 \setminus k$, the first homology of X_∞, $H_1(X_\infty)$, can be considered a module by noticing that it admits an action by $\mathbb{Z}G_{ab} = \mathbb{Z}[t^{\pm 1}]$. The module $H_1(X_\infty)$ is called the Alexander module and is itself an invariant of the knot k. Please see Chapter 6 of [**Lck**] for a development of this theory and pp. 115-118 to see how the matrix given here is indeed a presentation for this module. While the Alexander matrix here was constructed from an arbitrary group presentation, an alternative definition of the Alexander matrix is any presentation matrix for the Alexander module.

We now briefly develop the theory of elementary ideals, from which the Alexander polynomial is derived.

Definition: Given any $m \times n$ matrix A with entries in a commutative ring R, the *i-th elementary ideal* $E_i \subset R$ is the ideal generated by the determinants of all $(m-i) \times (m-i)$ submatrices of A (i.e., the $(m-i)$-minors of A). By convention, $E_i = 0$ if $(m-i) > n$ and $E_i = R$ if $(m-i) \leq 0$.

Since determinants can be expressed as a combination of smaller determinants via cofactor expansion, this definition clearly implies

$\mathbb{Z}[t^{\pm 1}]$ over $\mathbb{Z}[t, t^{-1}]$ since, formally, in the ring $\mathbb{Z}[t, t^{-1}]$, t and t^{-1} are just distinct variables. Thus, *a priori*, there's no reason $tt^{-1} = 1$. However, when we write $\mathbb{Z}[t^{\pm 1}]$, we identify $\mathbb{Z}[t^{\pm 1}] = \mathbb{Z}[t, t^{-1}]/\langle 1 - tt^{-1} \rangle$.

$$0 = E_0 \subset E_1 \subset \cdots \subset E_m \subset E_{m+1} = \cdots = R$$

From the invariant properties of the determinant, the following important theorem can be deduced:

Theorem 5.2: All elementary ideals $E_i \subset R$ are invariant under the following transformations of the matrix A:

T1. Permutation of rows and columns

T2. Multiplying a column or row by a unit from R

T3. Adding a multiple of one row (resp. column) to another row (resp. column)

T4. Replacing A with $\begin{pmatrix} A & \mathbf{0} \\ \mathbf{a} & x \end{pmatrix}$, where $\mathbf{0}$ is a column of zeroes, $\mathbf{a}$ is an arbitrary vector in R^n, and x is a unit in R.

T5. Adding or deleting a row of zeroes, i.e., replacing A with $\begin{pmatrix} A \\ \mathbf{0} \end{pmatrix}$

Proof: See [**CF**], Chapter VII, Sect. 4.

Remark: If A is considered as a presentation matrix of the Alexander module, any matrix which is equivalent to A under these transformations presents an isomorphic module. Again, see [**Lck**] for more details.

We now move towards a definition of the Alexander polynomial. Assume that k is a knot and let $G = \pi_1(\mathbb{R}^3 \setminus k) = \langle x_1, \ldots, x_n | r_1, \ldots, r_m \rangle$ be its knot group. (The Alexander matrix of k is then understood to be the Alexander matrix of G.) Note that $\mathbb{Z}[t^{\pm 1}]$ is a unique factorization domain (UFD), and therefore a principle ideal domain (PID). Thus, each ideal is generated by one element in $\mathbb{Z}[t^{\pm 1}]$.

Definition: Let k be a knot and A be its Alexander matrix. The *i-th Alexander polynomial of k*, denoted $\Delta_i(k)$, is the the generator of the smallest principle ideal containing the i-th elementary ideal of A. In other words, an element $\Delta_i(k) \subset \mathbb{Z}[t^{\pm 1}]$ such that $E_i \subset \langle \Delta_i(k) \rangle$.

Remarks:

- Note that this element is not quite unique, since if $E_i \subset \langle \Delta_i(k) \rangle$, then for any $n \in \mathbb{N}$, $\langle \pm t^{\pm n} \Delta_i(k) \rangle$ is also a principle ideal containing E_i.
- The "Alexander polynomial" with no qualifiers can be ambiguous, with different terminology being used by different authors. For this reason, it is important to note that it is most correct to specify the *first* or the 0th or the i-th Alexander polynomial, depending on what is being referred to.
- In order to compute Δ_i, one must simply:

  (1) find all the $(m-i)$-minors of the Alexander matrix

  (2) take the greatest common divisor of each of the $(m-i)$-minors. The resulting GCD is exactly Δ_i (up to multiplicative factors).

While we have finally arrived at the definition of Δ_i, we have yet to show that it is indeed an invariant. In order to do so, let us further develop the theory of Alexander matrices. We will say that two Alexander matrices are *equivalent* if they differ by a sequence of the transformations **T1** - **T5** from Theorem 5.2. (Note that this indeed an equivalence relation on the set of matrices with entries in a ring R.) In order to prove that Δ_i is invariant, we must make use of the following lemma (which is given without proof). The proof is based on the fact that two isomorphic group presentations differ by a sequence of *Tietze transformations.* [8]

Lemma 6.3: If P_1 and P_2 present isomorphic groups (i.e., differ by a sequence of Tietze transformations), then their corresponding Alexander matrices are equivalent (i.e., differ by a sequence of matrix transformations **T1** - **T5**).
Proof: Please see [**CF**], Chap. VII, Section 4 for details.

Theorem 6.4: Δ_i is a knot invariant (up to multiplication by $t^{\pm n}$).
Proof: The knot group G of a knot k is an invariant. Thus if P_1 and P_2 are two presentations for G coming from different knot diagrams of k, P_1 and P_2 are isomorphic. If A_{P_1} and A_{P_2} are

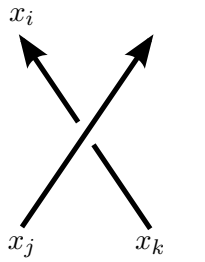
[8] See [**CF**], Chapter IV, Sect. 3 for more on Tietze transformations.

the Alexander matrices corresponding to the groups P_1 and P_2, respectively, then by the preceding lemma, they are equivalent matrices and differ only by transformations of the form **T1 - T5** from Theorem 5.2. By that theroem, elementary ideals of equivalent matrices are equal. Since Δ_i is defined in terms the i-th elementary ideal of the Alexander matrix, then Δ_i computed from A_{P_1} and A_{P_2} will coincide (up to multiplication by powers of $t^{\pm 1}$). Hence, Δ_i is an invariant of k (up to multiplication by powers of $t^{\pm 1}$).

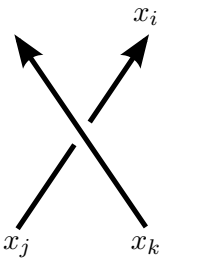
Calculating Δ_i via the Wirtinger Presentation

If we have a relatively nice presentation for our knot group, then it is easy to see from the definition of A that many entries will be zero. This is precisely what we see when we construct this matrix via the Wirtinger presentation. We will now explicitly compute the entries of the Alexander matrix induced by a crossing of an arbitrary knot diagram. This will not only allow us to efficiently compute Alexander polynomials, but we will later see how this derivation of the Alexander polynomial is related to more general results.

Recall that the Wirtinger presentation of the knot group of a knot k is given by $\langle x_1, \ldots, x_n | r_1, \ldots, r_n \rangle$, where each x_i corresponds to an arc of k and each r_i is determined by the i-th crossing according to the following rule (depending on whether the crossing is positive or negative):

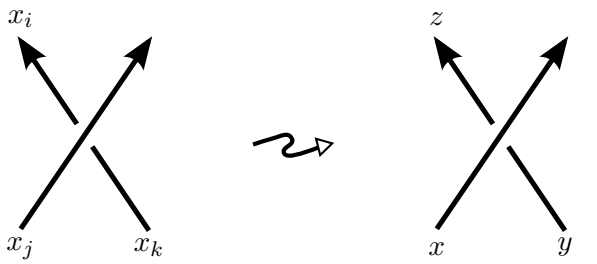


$r_i = x_i^{-1} x_j x_k x_j^{-1}$



$r_i = x_i^{-1} x_k^{-1} x_j x_k$

Let us examine the case of a positive crossing. For notational simplicity, relabel the arcs x_i, x_j, and x_k by z, x, and y, respectively.



Then the relator induced by this crossing will be $r = z^{-1}xyx^{-1}$. The entries in the Alexander matrix of the x, y, and z columns in the row corresponding to this crossing are given by

x	y	z
$\alpha\varphi(\partial r/\partial x)$	$\alpha\varphi(\partial r/\partial y)$	$\alpha\varphi(\partial r/\partial z)$

Note that the entries for any other column in this row (i.e., $\alpha\varphi(\partial r/\partial w)$ for $w \neq x, y$, or z) will be zero by the definition of the Fox derivative. For computational convenience, rewrite the relation r as $zx - xy = 0$. First, calculate the derivative portion of each of these entries:

$$\frac{\partial}{\partial x}(zx - xy) = \frac{\partial}{\partial x}(zx) - \frac{\partial}{\partial x}(xy) = \left(\frac{\partial z}{\partial x} + z\frac{\partial x}{\partial x}\right) - \left(\frac{\partial x}{\partial x} + x\frac{\partial y}{\partial x}\right) = z - 1$$

$$\frac{\partial}{\partial y}(zx - xy) = \frac{\partial}{\partial y}(zx) - \frac{\partial}{\partial y}(xy) = \left(\frac{\partial z}{\partial y} + z\frac{\partial x}{\partial y}\right) - \left(\frac{\partial x}{\partial y} + x\frac{\partial y}{\partial y}\right) = -x$$

$$\frac{\partial}{\partial z}(zx - xy) = \frac{\partial}{\partial z}(zx) - \frac{\partial}{\partial z}(xy) = \left(\frac{\partial z}{\partial z} + z\frac{\partial x}{\partial z}\right) - \left(\frac{\partial x}{\partial z} + x\frac{\partial y}{\partial z}\right) = 1$$

Applying φ (which simply sends the relators to zero) to each of these entries does not change them. Lastly, as mentioned before, applying the abelianization map α sends each of the generators, x, y, and z to a single generator t. Thus, the entries in the above table can be written explicitly as

x	y	z
$t-1$	$-t$	1

These entries in the Alexander matrix correspond to the relation in the Alexander module given

by

$$z = yt + (1-t)x$$

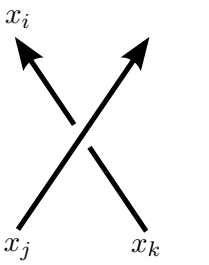
It is an easy calculation left to the reader verify that the entries corresponding to a negative crossing are given by

x	y	z
-1	$1-t$	t

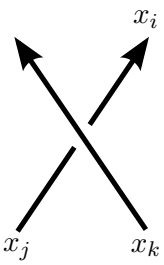
or equivalently $\quad x = tz + (1-t)y$

Summarizing Remarks

Let us recap the work we have done thus far. We have defined Δ_i (the i-the Alexander polynomial) by analyzing the knot group. This is useful because (1) we already know the knot group is an invariant and (2) Δ_i can be computed from any presentation of the knot group. However, notice that we really only need a presentation matrix for the Alexander module (called the Alexander matrix) in order to computer Δ_i—in other words, the computation of Δ_i via fundamental groups is instructive, but not the most efficient method. By considering the Wirtinger presentation of a knot group, we were able to find the entries in the Alexander matrix for an arbitrary crossing on a knot diagram. Hence, this whole process can be streamlined. We can simply consider the module generated by the arcs of the knot, with the following relations given by each crossing, depending on whether the crossings are positive or negative.



	x_i	x_j	x_k
r_i	1	$t-1$	$-t$



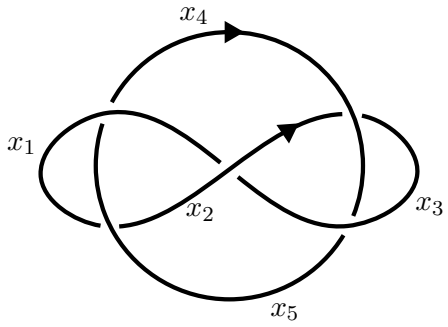
	x_i	x_j	x_k
r_i	t	-1	$1-t$

If a knot k has n crossings, we can construct an $n \times n$ matrix with entries determined by the rules above for each crossing. Even though this process involves no mention of knot groups or the Fox

calculus, its end result is the exact same Alexander matrix obtained in the earlier discussion. Once this matrix has been found, the computation of Δ_i proceeds as before.

Example

Let us compute the Alexander polynomial of the Whitehead link, illustrated below.[9]



We can construct the 5×5 Alexander matrix of this link according the rules above.

	x_1	x_2	x_3	x_4	x_5
r_1	1	$t-1$	$-t$	0	0
r_2	-1	t	0	0	$1-t$
r_3	0	$-t$	1	$t-1$	0
r_4	$1-t$	0	0	t	-1
r_5	0	0	$t-1$	$-t$	1

The first minors of this matrix are[10]

$$\begin{pmatrix} (-1+t)^3t & (1-t)(-1+t)^2t & (-1+t)^3t & -(-1+t)^3t & (-1+t)^3t \\ (1-t)(-1+t)^2 & (-1+t)^3 & -(-1+t)^3 & (-1+t)^3 & -(-1+t)^3 \\ (-1+t)^3t & (1-t)(-1+t)^2t & (-1+t)^3t & -(-1+t)^3t & (-1+t)^3t \\ (1-t)(-1+t)^2 & (-1+t)^3 & -(-1+t)^3 & (-1+t)^3 & -(-1+t)^3 \\ (-1+t)^3t & (1-t)(-1+t)^2t & (-1+t)^3t & -(-1+t)^3t & (-1+t)^3t \end{pmatrix}$$

[9] Recall that in the original definition of Δ via the Fox calculus, we used the fact that the abelianization of the knot group is infinite cyclic. This is actually not true for *links*, like the one considered here. Nonetheless, we can still perform the same calculations, sending all generators of the link group to a single generator t. However, there is a way to account for this difference between knots and links; this subject is addressed in the next section.

[10] The calculations in this section were performed using Mathematica. The command `Flatten[Minors[ ]]` with matrix input produces all first minors of the matrix in list form. One can copy this result (without the braces { }) into the command `PolynomialGCD[ ]` to obtain the greatest divisor of these minors.

Taking the greatest divisor of these polynomials gives us $\Delta_1(t) = (t-1)^2 = t^2 - 2t + 1$.

5.3 Multivariable Alexander Polynomial for Classical Links

It is worth noting that the theory of Alexander polynomials for links of multiple components can be amplified to provide more powerful invariants (cf. [**BZ**], Ch. 9D; [**Lck**], p. 119). In the case of an l-component link L, the abelianization of its link group is not just infinite cyclic, but rather the direct sum of l copies of the infinite cyclic group.

$$\pi_1(\mathbb{R}^3 \setminus L) = \overset{l \text{ times}}{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}} = \langle t_1, \ldots, t_l \rangle_{ab}$$

Each generator t_i is the generator of the knot group of the i-th component of L. In our construction of the Alexander matrix, note that when we applied Fox derivative to the relators of our link group, we obtained the relations

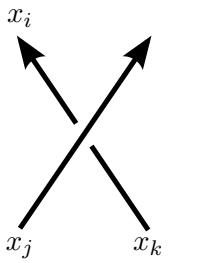
x	y	z
$z-1$	$-x$	1

However, upon abelianizing our module, each of the generators x and z were sent to a single generator t. If we were, instead, to send the variables x and z to the generators of the abelianization of the their respective components, we would obtain the relations

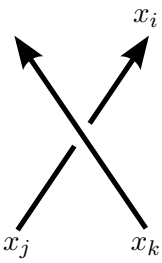
$$z = yt_o + (1 - t_u)x$$

where t_u corresponds to the generator of the component in the *understrand* of the crossing, and t_o corresponds to the generator of the component of the *overstrand* in the crossing (and analogous relation for negative crossings). With this extra care, our Alexander matrix would then present a $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_l^{\pm 1}]$-module from which we could consider elementary ideals and compute generators of these ideals in the same way as before. The resulting polynomial, which we will denote by $\Delta_i(t_1, \ldots, t_l)$ will indeed be an invariant (up to multiplication by units in $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_l^{\pm 1}]$). We call the module constructed in this process the *multivariable Alexander module* and the polynomial derived the *multivariable Alexander polynomial*. We will revisit this multivariable construction when we have developed a more robust theory for virtual knots.

Entries in the i-th row of the multivariable Alexander matrix



	x_i	x_j	x_k
r_i	1	$t_u - 1$	$-t_o$



	x_i	x_j	x_k
r_i	t_o	-1	$1 - t_u$

It is not difficult to adjust the matrix calculated in the example above involving the Whitehead link according to these new rule. Doing so produces the multivariable Alexander polynomial of the Whitehead link as $\Delta_1(t_1, t_2) = (t_1 - 1)(t_2 - 1)$.

6 EXTENDING INVARIANTS TO VIRTUAL KNOTS

Let us now consider invariants of virtual knots in earnest. We will first extend some classical invariants and proceed to construct a new invariant by generalizing the approach taken in the classical case.

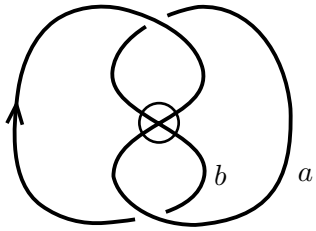
6.1 Virtual Knot Group and Alexander Modules of Virtual Knots

Note how the Wirtinger presentation of the knot group is based solely on local relations between arcs at crossing points of a knot diagram. This construction makes no reference to the ambient topology of $\mathbb{R}^3$, though the group presented is indeed isomorphic to $\mathbb{R}^3 \setminus k$. Note that in virtual knot theory, the topological interpretation of the complement of a virtual knot is not so easy to conceptualize. However, given a construction of the knot group which depends only on local crossing relations, there is hope that this construction can be extended to virtual knots. Recall that the idea behind virtual crossings is that they are not really there. Thus, a reasonable attempt to construct the "knot group" of a virtual knot is to carry out Wirtinger's construction on a virtual knot diagram, simply by ignoring virtual crossings. Even though the group constructed in this manner obviously won't be the fundamental group of the knot's ambient space, it still has the potential to be an invariant.

This approach was explored by Kauffman in his original paper on virtual knot theory [**Kau**]. He found that the Wirtinger presentation does, indeed, allow us to construct *virtual knot groups* from virtual diagrams by ignoring virtual crossings. A natural question, then, is whether the construction of the Alexander polynomial described in the previous section generalizes to virtual knots as well. Sawollek [**Saw**] addressed this question and answered it in the affirmative. Thus, one way to define the Alexander module and polynomial for virtual knots via the virtual knot group. Of course, in doing so, this virtual Alexander module will not have an equivalent topological definition as it does in the classical case (i.e., derived from the first homology of the infinite cover of the knot complement). Nonetheless, we will explore this formulation of the Alexander polynomial for virtual knots.

Insufficiency of Classical Approach

Let us consider the Alexander polynomial of a simple virtual knot. We will find, unfortunately, that this formulation of the Alexander polynomial is insufficient for detecting the non-triviality of the so-called virtual trefoil.



Denote this knot by k. Note that if we ignore the virtual crossing in the diagram, k will have only two arcs (a and b above). Let us construct the Alexander module of k in the same manner we would in the classical case. This module will have generators a and b with two relations: $A(k) = \langle a, b \mid r_1, r_2 \rangle$. The relation induced by the bottom crossing will be given by $b = at + (1-t)a$. However, note that this relation by itself is enough to trivialize $A(k)$, since it is equivalent to $b = a$. Thus $A(k)$ does not distinguish k from the unknot. Note that in the classical world, any knot with just two arcs must be trivial (since it will have only one crossing). Intuitively, the knot above seems like it should be non-trivial. Thus it appears that, in the virtual theory, more complicated knots may still only have two arcs. (After we develop some theory, we will indeed demonstrate that the virtual trefoil is non-trivial).

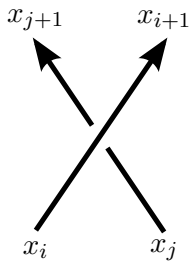
The previous example motivates us to come up with stronger invariants which can detect the non-triviality of knots like the virtual trefoil. The remainder of this section is dedicated to providing an intuitive way of explaining how we might do so.

6.2 Developing a Stronger Invariant

Semi-arcs

Given the phenomenon that we saw in the virtual trefoil (i.e., that a virtual knot with two arcs can be trivial), we are motivated to reconsider what it means to be an arc in the virtual sense. This leads us to consider the notion of **semi-arcs** (or *edges*). Semi-arcs are designated as portions

of a knot diagram from one classical crossing to the next (regardless of whether a strand is over or under), ignoring any virtual crossings. This is equivalent to dividing each arc at every point it crosses over another strand. Notice any classical crossing will involve four semi-arcs and a diagram with n classical crossings will have $2n$ semi-arcs.



Classical crossing with semi-arcs labeled

Note that, with our new definition, any virtual knot with just two semi-arcs is certainly trivial.

Constructing an Invariant Module from Semi-Arcs

In the classical case, we could construct the Alexander module as the module generated by the arcs of a knot with relations induced by the crossings. Let us try to generalize this kind of construction to virtual knots, except using semi-arcs as generators instead. However, instead of having a specific set of relations specified, we will consider a more general way to define relations between the semi-arcs. We can do this by thinking of the semi-arcs going into a crossing being acted on by an arbitrary operator whose outputs determine the labeling of the outgoing semi-arcs.

To make this more precise, let k be a virtual knot with n classical crossings and semi-arcs labeled $x_1, \ldots, x_{2n}$. Let R be an arbitrary ring and let $\mathcal{M}$ be the R-module generated $x_1, \ldots x_{2n}$ with relations determined in the following manner: Let $S : \mathcal{M}^2 \to \mathcal{M}^2$ and $S' : \mathcal{M}^2 \to \mathcal{M}^2$ be arbitrary operators. Each crossing will have two incoming semi-arcs, x_i and x_j, and two outgoing semi-arcs, x_{j+1} and x_{i+1}. For every positive crossing on the knot diagram, impose the two relations determined by $(x_{j+1}, x_{i+1}) = S(x_i, x_j)$ and for each negative crossing impose the relations determined by $(x_{j+1}, x_{i+1}) = S'(x_i, x_j)$.

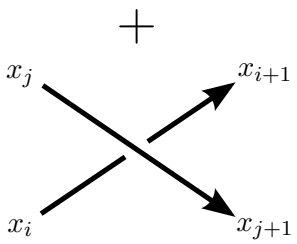
For simplicity in this section, we will assume two things: (1) that R is a commutative ring and (2)

that S and S' are linear operators. (These assumptions are not necessary, but it allows for the most transparent and direct motivation of the more general cases. For more on this, see the remarks regarding the work of [**FJK**] and [**BF**] in the next section.)

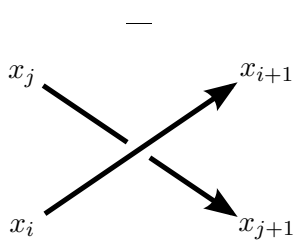
With these assumptions, we will have $S, S' \in \mathrm{Mat}_{2\times 2}(R)$ which we can represent by

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad\qquad S' = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}$$

We can now summarize the construction of $\mathcal{M}$ in the following way: Given a virtual knot k with n classical crossings and semi-arcs labeled $x_1, \ldots, x_{2n}$, we let $\mathcal{M}$ be the module generated by $x_1, \ldots x_{2n}$ with the relations given below on left for each positive crossing of k and on the right for each negative crossing of k.



$$(x_{j+1}, x_{i+1}) = S(x_i, x_j)$$
$$x_{j+1} = Ax_i + Bx_j$$
$$x_{i+1} = Cx_i + Dx_j$$



$$(x_{j+1}, x_{i+1}) = S'(x_i, x_j)$$
$$x_{j+1} = A'x_i + B'x_j$$
$$x_{i+1} = C'x_i + D'x_j$$

From this, we can see that $\mathcal{M}$ will be determined by a $2n \times 2n$ presentation matrix; $2n$ columns corresponding to each semi-arc and $2n$ rows corresponding to the two relations induced by each of the n crossings. *A priori*, there is no reason why $\mathcal{M}$ should be invariant. However, we can try to determine what conditions must hold for the operators S and S' if they are to make $\mathcal{M}$ an invariant module. We proceed by considering the effect on $\mathcal{M}$ by each of the generalized Reidemeister moves.

Determining Sufficient Conditions for Invariance of $\mathcal{M}$

Virtual Reidemeister Moves

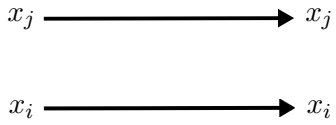
Since the labeling of semi-arcs (and hence the generators of $\mathcal{M}$) are unchanged by the virtual Reidemeister moves, $\mathcal{M}$ is automatically invariant under any of these moves (including the semi-virtual move, which similarly does not change the labeling of semi-arcs). Hence, we must only make sure that $\mathcal{M}$ is unchanged by the classical three Reidemeister moves.

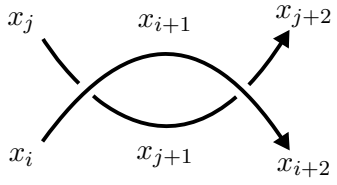
As we move on to consider the classical Reidemeister moves, each section will consist of two parts: (1) determining conditions on S that are implied by insisting $\mathcal{M}$ be invariant and (2) proving that $\mathcal{M}$ is indeed invariant under these conditions (and perhaps some additional assumptions which will make calculations more manageable). For reasons which should become apparent, we will consider the classical Reidemeister moves in the following order: RII, RIII, RI.

Classical Type II Reidemeister Move (RII)

Let M and M' be presentation matrices for $\mathcal{M}$, computed from a knot diagram differing only by one Reidemeister move. Recall that in Theorem 5.2, we gave sufficient conditions to determine if M and M' present isomorphic modules. Hence, in order for $\mathcal{M}$ to be invariant under the Reidemeister moves, the operators S and S' must be defined in such a way so that M and M' differ by a series of the transformations **T1**-**T5** from Theorem 5.2.

Let us first determine conditions on S and S' so that $\mathcal{M}$ is unchanged by the second Reidemeister move.[11]





We must first require that the labels on the outgoing strands match in each figure above, i.e., that $(x_{i+2}.x_{j+2}) = (x_i.x_j)$. This insures that any other relations involving x_i and x_j will be unchanged by moving from the diagram on the left to the right (and visa versa). Note, however, that since

[11] For full rigor, one must pay particularly close attention to the orientations of the strands in each of the Reidemeister moves. If an invariant only holds for a Reidemeister move of the orientation given in this example, this is not sufficient to conclude that it holds for all possible orientations of the strands in an RII move. This issue is addressed in Section 9.

$(x_{i+2}, x_{j+2}) = S(x_{j+1}, x_{i+1})$ and $(x_{j+1}, x_{i+1}) = S'(x_i, x_j)$, this is the same as requiring

$$(x_i.x_j) = (x_{i+2}, x_{j+2}) = S(S'(x_i, x_j))$$

Similarly, if $\mathcal{M}$ is to remain unchanged by the following move

we must require that

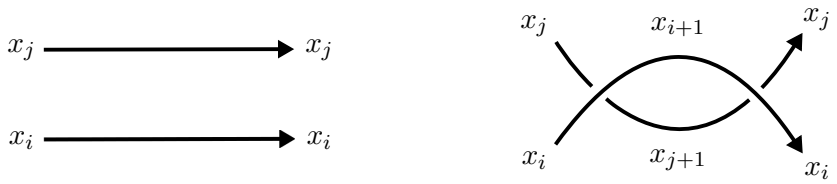
$$(x_i.x_j) = (x_{i+2}, x_{j+2}) = S'(S(x_i, x_j))$$

Combining each of these conditions, we see precisely that $SS' = S'S = id_{\mathcal{M}^2}$. In other words, $S' = S^{-1}$. Thus, instead of having two separate operators for positive and negative crossings, the type II Reidemeister move requires that there be only one operator, which acts inversely between the arcs of positive and negative crossings. Note that we have been thinking about the action of S (resp. S') as the left-to-right action across positive (resp. negative) crossings. Having determined that $S' = S^{-1}$, we write all relations in terms of just S. Specifically, instead of thinking of S^{-1} as the left-to-right action across negative crossings, we will instead think of S as the *right-to-left* action across negative crossings. Hence, the rules for writing crossing relations can be reduced to the following diagram:

$$(z, w) = S(x, y)$$
$$z = Ax + By$$
$$w = Cx + Dy$$

We now move on to consider how $\mathcal{M}$ is affected by this Reidemeister move. The invertibility of S

implies two things: (1) $\det S \neq 0$ and is invertible, i.e., that $AD - BC$ is a unit and (2) that the incoming/outgoing semi-arcs will have the same labels in each figure. We can redraw the figures as



Note, however, that we have yet to show that the module $\mathcal{M}$ is unchanged by the Reidemeister move. Specifically, we must show the presentation matrix for the module on the left is equivalent (in the sense of Theorem 5.2) to that of the matrix on the right.

Let M be the presentation matrix for a knot diagram representing the figure on the left. Note that in passing from the figure on the left to the right, two new generators are added and only two new relations are added (since the relations induced by the left crossing and the right crossing will be identical). This will change the matrix M to one of the form

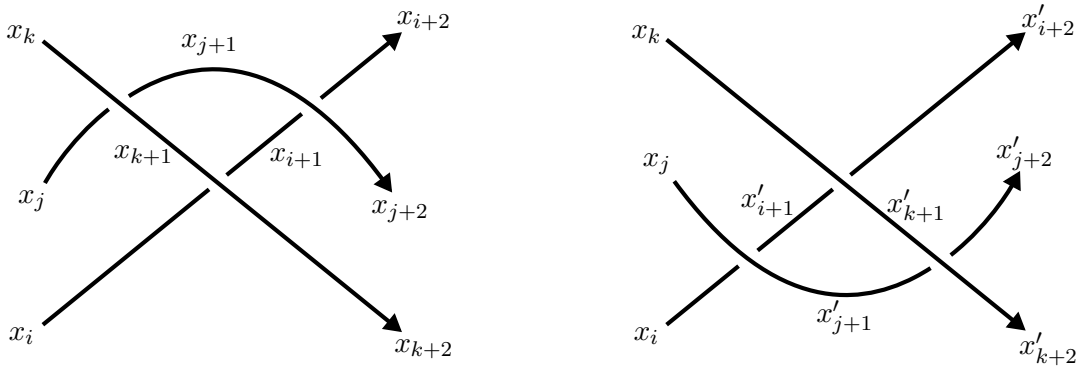
$$\begin{pmatrix} & M & & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & -1 & A & B \\ \mathbf{0} & -1 & 0 & C & D \end{pmatrix}$$

where $\mathbf{0}$ represents either a row or column of zeroes. The columns containing the -1's have been arranged so that these correspond precisely to x_i and x_j and the columns on the far right correspond to the new generators x_{i+1} and x_{j+1}. Note that if one of either A and D or B and C are units in R, this new matrix is equivalent to M via two **T4** transformations.

In conclusion, we have shown that $\mathcal{M}$ is invariant under RII under the assumptions that S is an invertible operator, $S' = S^{-1}$, and either A and D or B and C are units in R. At the cost of losing a small degree of generality, let us assume that B and C are units from here on out.

Classical Type III Reidemeister Move (RIII)

Let us try to derive a similar condition on S which guarantee $\mathcal{M}$ will be invariant under RIII. Below is a diagram of an RIII move with all semi-arcs labeled.



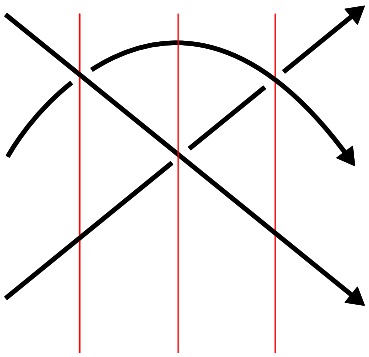
The semi-arcs on the right are designated with a prime since, *a priori*, there's no reason that they should have the same labels as the semi-arcs in the diagram on the left. Again, we must require that the labels on the outgoing semi-arcs coincide:

$$(x_{i+2}, x_{j+2}, x_{k+2}) = (x'_{i+2}, x'_{j+2}, x'_{k+2})$$

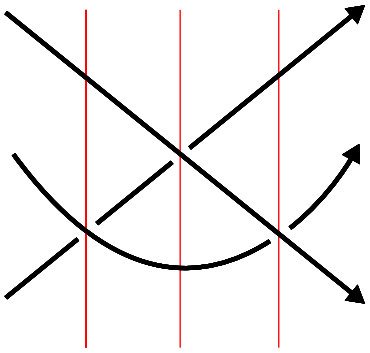
Because there are three arcs involved in this instance, it would be nice if we could express the action of S in terms of a three-coordinate function in x_i, x_j, and x_k. To do this, we introduce the notation

$$\begin{aligned} &(S \times id) : \mathcal{M}^3 \to \mathcal{M}^3 \quad \text{and} \quad (id \times S) : \mathcal{M}^3 \to \mathcal{M}^3 \\ &(a,b,c) \mapsto (S(a,b),c) \qquad\qquad (a,b,c) \mapsto (a, S(a,b)) \end{aligned}$$

This allows us to express the left-to-right action the crossings have on the three semi-arcs in the Reidemeister diagram. Note that each crossing in this diagram is positive, so our relations will only involve S (though the functions $(S^{-1} \times id)$ and $(id \times S^{-1})$ can be just as well defined). The diagram below illustrates this concept most directly.



$(id \times S)(S \times id)(id \times S)$



$(S \times id)(id \times S)(S \times id)$

When looking at this picture, the labels of the incoming semi-arcs of each figure are the input to some function which will act on them at each red line. The three semi-arcs immediately to left of a red line can be thought of as a three-tuple input and the three arcs immediately to the right of the red line as a three-tuple output. It's not hard to see that the function acting as you cross each line from left to right is exactly the function listed below the line.

$$(x_i, x_j, x_k)(id \times S)(S \times id)(id \times S) \qquad \text{and} \qquad (x_i, x_j, x_k)(S \times id)(id \times S)(S \times id)$$

Here, we point out that we are writing function composition in a left-to-right ordering (not the usual right-to-left). In this case, because the factors in the composition are symmetric, it doesn't matter whether we simply concatenate the functions as in the diagram above or list them in usual right-to-left composition order. However, we make this remark because the ordering will play a role in Section 8.

From this, we see that, in order for the outgoing semi-arcs to have the same label in each figure, S must satisfy the following equation

$$(id \times S)(S \times id)(id \times S) = (S \times id)(id \times S)(S \times id)$$

This equation is often referred to as the *Yang-Baxter equation*, due to its relationship with equations that were originally derived in the context of statistical mechanics, but have also arisen in several other disciplines (such as differential equations, quantum field theory, and quantum groups; see [**PA**] and [**Dr**] for more on these equations.)

Note that the matrix representation of $S \times id$ and $id \times S$ are given by

$$S \times id = \begin{pmatrix} A & B & 0 \\ C & D & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad id \times S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & A & B \\ 0 & C & D \end{pmatrix}$$

The Yang-Baxter equations then become

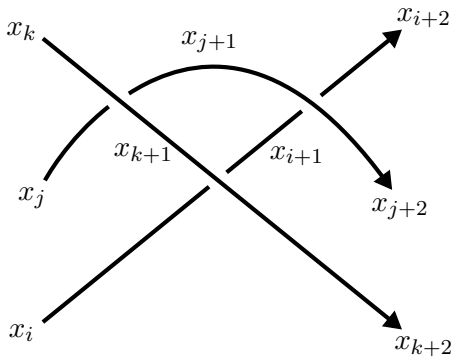
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & A & B \\ 0 & C & D \end{pmatrix} \begin{pmatrix} A & B & 0 \\ C & D & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & A & B \\ 0 & C & D \end{pmatrix} =$$

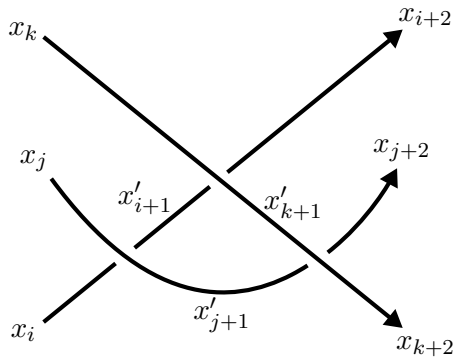
$$\begin{pmatrix} A & B & 0 \\ C & D & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & A & B \\ 0 & C & D \end{pmatrix} \begin{pmatrix} A & B & 0 \\ C & D & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} A & AB & B^2 \\ AC & A^2D + BC & ABD + BD \\ C^2 & ACD + CD & D^2 + BCD \end{pmatrix} = \begin{pmatrix} A^2 + ABC & AD + ABD & B^2 \\ AC + ACD & AD^2 + BC & BD \\ C^2 & CD & D \end{pmatrix}$$

As can be seen, this implies seven nontrivial identities among the elements A, B, C, D. We will revisit these equations momentarily. Let us first consider how $\mathcal{M}$ is affected by a type III move.

We have already shown that the as long as S satisfies the Yang-Baxter equations, then the outgoing labels will coincide in each of the following two figures.





Hence, we need only consider how the presentation matrix changes between the entries involving the generators x_{i+1}, x_{j+1}, and x_{k+1} and x'_{i+1}, x'_{j+1}, and x'_{k+1}. The relations involving these generators can be deduced from the two left-most crossings in each diagram. For the figure on the left, the relations induced by these crossings will be given by $S(x_j, x_k) = (x_{k+1}, x_{j+1})$ and $S(x_i, x_{k+1}) = (x_{k+2}, x_{i+1})$, which imply the following equations

$$\begin{array}{llll} (1) & Ax_j + Bx_k = x_{k+1} & \text{and} & (3) \quad Ax_i + Bx_{k+1} = x_{k+2} \\ (2) & Cx_j + Dx_k = x_{j+1} & & (4) \quad Cx_i + Dx_{k+1} = x_{i+1} \end{array}$$

Equations (3) and (4) can be rewritten using (1) so they only involve the initial generators x_i, x_j and x_k.

For the figure on the right, the relations will be $S(x_i, x_j) = (x'_{j+1}, x'_{i+1})$ and $S(x'_{i+1}, x_k) = (x'_{k+1}, x_{i+2})$ or more explicitly

$$\begin{array}{llll} (1') & Ax_i + Bx_j = x'_{j+1} & \text{and} & (3') \quad Ax'_{i+1} + Bx_k = x'_{k+1} \\ (2') & Cx_j + Dx_k = x'_{i+1} & & (4') \quad Cx'_{i+1} + Dx_k = x_{i+2} \end{array}$$

Again, equations (3') and (4') can be simplified using equation (2') so only the initial generators are in each relation. It is not hard to see, then, that the presentation matrices of the each of the diagrams will have the form

	x_i	x_j	x_k	x_{i+1}	x_{j+1}	x_{k+1}
		M		$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
$\mathbf{0}$	C	AD	BD	-1	0	0
$\mathbf{0}$	0	C	D	0	-1	0
$\mathbf{0}$	0	A	B	0	0	-1

	x_i	x_j	x_k	x'_{i+1}	x'_{j+1}	x'_{k+1}
		M		$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
$\mathbf{0}$	C	D	0	-1	0	0
$\mathbf{0}$	A	B	0	0	-1	0
$\mathbf{0}$	AC	AD	B	0	0	-1

where M is the portion of the presentation matrix unaffected by this move. (The relations (3) and (4') are unchanged between the diagrams by the Yang-Baxter equations, and hence are contained in M.) From this we can see that the equivalence of these presentation matrices reduces to the

equivalence of the matrices

$$\begin{pmatrix} C & AD & BD \\ 0 & C & D \\ 0 & A & B \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} C & D & 0 \\ A & B & 0 \\ AC & AD & B \end{pmatrix}$$

Call these X and Y, respectively. We will prove that Y can be taken to X through the transformations **T1-T5**. Recall that we assumed B and C are units and we also assumed our operator S is invertible, which guaranteed that its determinant $AD - BC$ is a unit in R. We make one last additional assumption which greatly simplifies the following calculation: that D is also a unit.[12] Of course, the product of units is also a unit.

Proposition 6.1: Under the assumptions just mentioned, the matrices X and Y are equivalent (in the sense of Theorem 5.2). Hence the matrices calculated from a knot differing by only a type III Reidemeister move present isomorphic modules.

Proof: We find a sequence of transformations **T1-T5** that take Y to X.

$$\begin{pmatrix} C & D & 0 \\ A & B & 0 \\ AC & AD & B \end{pmatrix} \xrightarrow{\textbf{T1}} \text{permute rows and columns} \begin{pmatrix} B & AC & AD \\ 0 & C & D \\ 0 & A & B \end{pmatrix} \xrightarrow{\textbf{T2}} \text{multiply by } B^{-1}(AD-BC)$$

$$\begin{pmatrix} AD-BC & AC & AD \\ 0 & C & D \\ 0 & A & B \end{pmatrix} \xrightarrow{\textbf{T3}} \text{subtract column 1 from 3} \begin{pmatrix} AD-BC & AC & BC \\ 0 & C & D \\ 0 & A & B \end{pmatrix} \xrightarrow{\textbf{T2}} \text{multiply by } C^{-1}D$$

$$\begin{pmatrix} (AD-BC)C^{-1}D & AD & BD \\ 0 & C & D \\ 0 & A & B \end{pmatrix} \xrightarrow{\textbf{T3}} \text{multiply by } ((AD-BC)C^{-1}D)^{-1}C \begin{pmatrix} C & AD & BD \\ 0 & C & D \\ 0 & A & B \end{pmatrix}$$

Thus, the presentation matrices are indeed equivalent which proves the invariance of $\mathcal{M}$ under a type III Reidemeister move. □

Taking the Yang-Baxter Equations a Step Further: Let's revisit the relations implied by

[12] Similar matrix transformations could be made if we were to assume that A were in a unit instead of D. We made an arbitrary choice between them, which will have a minor consequence in the next section regarding the Yang-Baxter equations—the next footnote will mention this consequence.

the Yang-Baxter equation. Equating entries in each matrix of

$$\begin{pmatrix} A & AB & B^2 \\ AC & A^2D+BC & ABD+BD \\ C^2 & ACD+CD & D^2+BCD \end{pmatrix} = \begin{pmatrix} A^2+ABC & AD+ABD & B^2 \\ AC+ACD & AD^2+BC & BD \\ C^2 & CD & D \end{pmatrix}$$

we have the seven equations

$$\begin{array}{ccc} (1)\quad A = A^2+ABC & (2)\quad AB = AD+ABD & (3)\quad AC = AC+ACD \\ (4)\quad A^2D+BC = AD^2+BC & (5)\quad ABD+BD = BD & (6)\quad ACD+CD = CD \\ & (7)\quad D^2+BCD = D & \end{array}$$

Note equations (3) and (5) reduce to

$$ABD = 0 \qquad \text{and} \qquad ACD = 0$$

Since we have already assumed that B, C and D are units (i.e., non-zero), it follows that $A = 0$. After eliminating all terms with A, the only non-trivial relation remaining is $D^2 + BCD = D$, which reduces to $D = 1 - BC$. Thus, we have arrived at a surprisingly simple form of the operator S:

$$S = \begin{pmatrix} 0 & B \\ C & 1-BC \end{pmatrix}$$

where B and C are invertible elements in any commutative ring R.[13] Under the assumptions we have made about the entries of S, we have found that this is the only possible form it can take. From now on, S will refer to exactly the matrix of this form. Because we have made several assumptions up to this point, we state our previous work as a theorem for easy reference and subsequently proceed to consider the last Reidemeister move.

[13]Note that if we had assumed A was a unit instead of D, we would have similarly derived $S = \begin{pmatrix} 1-BC & B \\ C & 0 \end{pmatrix}$

Theorem 6.2. Let R be a commutative ring and $\mathcal{M}$ be an R-module. Let $S \in Aut(\mathcal{M})$ be an invertible linear operator with matrix representation

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

Furthermore let B and C be units and S satisfy the Yang-Baxer equations $(id \times S)(S \times id)(id \times S) = (S \times id)(id \times S)(S \times id)$. If we assume D is also a unit then

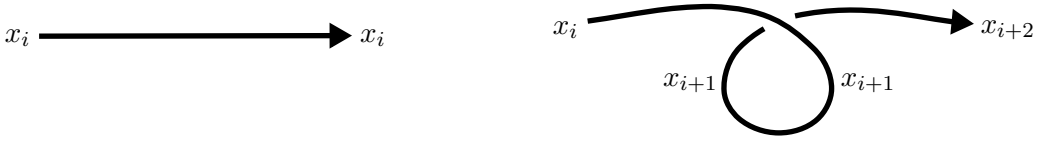
$$S = \begin{pmatrix} 0 & B \\ C & 1 - BC \end{pmatrix}$$

If we instead assume A is a unit then

$$S = \begin{pmatrix} 1 - BC & B \\ C & 0 \end{pmatrix}$$

Classical Type I Reidemeister Move (RI)

We proceed as in the previous two cases.



The relation added by the crossing in the figure on the right is $(x_{i+1}, x_{i+2}) = S(x_{i+1}, x_i)$. As in both the previous cases, in order for $\mathcal{M}$ to be invariant, we want to require that $x_{i+2} = x_i$. However, by using the form of S we previously derived, we will see that this requirement does not force us to assume anything else about our operator S. In other words—under all the assumptions we have already made—the invariance of $\mathcal{M}$ under the Reidemeister II and III moves is sufficient to guarantee invariance under RI.

We see this by noting the relations implied by $(x_{i+1}, x_{i+2}) = S(x_{i+1}, x_i)$ become

$$x_{i+1} = Bx_i \qquad \text{and} \qquad x_{i+2} = Cx_{i+1} + (1 - BC)x_i$$

After substituting the first equation into the second, we see

$$x_{i+2} = CBx_i + (1 - BC)x_i = x_i$$

from which we see that the outgoing strand will indeed have the same label as the incoming one in each diagram. Taking this into account, we can rewrite the relations with x_i in place of x_{i+2}. This reduces the two relations to the single one $x_{i+1} = Bx_i$.

As for the invariance of $\mathcal{M}$, if M is the presentation matrix coming from the diagram on the left, the matrix coming from the diagram on the right will have the form

$$\begin{pmatrix} & M & \mathbf{0} \\ \mathbf{0} & B & -1 \end{pmatrix}$$

which is clearly equivalent to M via a **T4** transformation.

Summarizing Results Regarding S and $\mathcal{M}$

The last section has been somewhat longwinded, but not without a powerful result. We summarize the work just done in the following theorem, which is proven by the previous discussion (save for a small caveat; see remark after).

Theorem 6.3. Let k be a virtual knot (or link) with n classical crossings and semi-arcs labeled $x_1, \ldots, x_{2n}$. Let R be a commutative ring with units B and C. Let $\mathcal{M}_k$ be the R-module generated by $x_1, \ldots, x_{2n}$ modulo the following relations determined by classical crossings:

$$z = By$$
$$w = Cx + (1 - BC)y$$

where x, y, z, and w are appropriately labeled elements in $\{x_1, \ldots, x_{2n}\}$, depending on whether the crossing is positive or negative. Then $\mathcal{M}_k$ is a virtual knot invariant of k (i.e., unchanged by classical, virtual, or semi-virtual Reidemeister moves).

It can be seen that $\mathcal{M}$ has a $2n \times 2n$ presentation matrix from which elementary ideals can be derived—it is an immediate corollary that the chain of elementary ideals is also invariant.

Remarks

- The caveat about this theorem is that we did not check all the possible orientations of arcs in each of the Reidemeister diagrams. In his paper [**Pol**], Michael Polyak demonstrated that arc orientation is quite significant. If all permutations of arcs are taken into account, there are not three but 16 possible Reidemeister moves one may encounter in transforming a knot diagram. Some of these moves are implied by others, but the full set 16 moves is actually not generated by the moves mentioned in this section. We discuss this issue as it pertains to the work in this paper in Section 9.

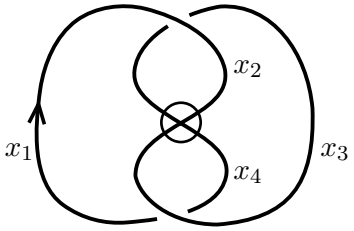
- The entirety of this section perhaps culminates in this observation: Because the elementary ideals of this module are invariant, it would be nice if—in analogy with the derivation of Alexander polynomials—we could find principal ideals containing these elementary ideals. This motivates us to consider the case when R is a UFD. Suppose $R = \mathbb{Z}[s^{\pm 1}, t^{\pm 1}]$ with $B = s$ and $C = t$. In this case, we call the module $\mathcal{M}$ defined by the theorem the *generalized Alexander module*. This name is justified since by setting $s = 1$, we get exactly the Alexander module found in section 6. Note that this shows, in some sense, the (generalized) Alexander module is essentially the only knot module over a commutative ring defined by linear relations between generators. By considering the generators of the smallest principle ideals containing

each elementary ideal of this module, we can define the *generalized Alexander polynomials* in the same way we did for classical knots. These will similarly be knot invariants up to multiplication by powers of $s^m t^n$, for $m, n \in \mathbb{Z}$. We will denote these by $\Delta_i(s,t)$.

- While I have yet to find an exposition as detailed and meticulous as the work done in the this section, this work is not original. Actually, there have been several authors who have independently arrived at essentially the same results (see the next section). I hope the work done in this paper, in addition to filling in some of the details (particularly those relating to Polyak's work), can be a useful pedagogical tool for illustrating a natural approach to generalizing knot invariants and also how a mathematician might arrive at new results.

6.3 Example

It is a well known result that $\Delta_0(t)$, the 0th Alexander polynomial is trivial for any classical knot ([**CF**], Chapter VII, 3.5). However, as we will see, this need not be the case for virtual knots. We use the 0th generalized Alexander polynomial to show that the virtual trefoil considered at the beginning of this section is indeed non-trivial.



The two classical crossings in the diagram above induce the following 4 relations between the semi-arcs (according to the rules in Theorem 6.3, with $B = s$ and $C = t$):

$$x_1 = sx_2 \qquad x_2 = sx_3$$
$$x_3 = tx_4 + (1-st)x_2 \qquad x_4 = tx_1 + (1-st)x_3$$

In matrix form, these relations become

$$\begin{pmatrix} -1 & s & 0 & 0 \\ 0 & (1-st) & -1 & t \\ 0 & -1 & s & 0 \\ t & 0 & (1-st) & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \mathbf{0}$$

The determinant of these equations gives us $\Delta_0(s,t) = (s-1)(t-1)(st-1) = -1+s+t-s^2t-st^2+s^2t^2$. Thus, though the regular Alexander polynomial was not enough to detect the virtual trefoil, the 0th generalized Alexander polynomial does indeed indicate that this knot is non-trivial.

7 SURVEY OF THE LITERATURE ON VIRTUAL ALEXANDER INVARIANTS

This section will provide an overview of the most important papers on the topic of the Alexander polynomial for virtual knots.

7.1 General Overview

Several authors have defined extensions of the Alexander polynomial to virtual knots. Following is a chronological list of some of the most relevant articles (and their authors) relating to the Alexander polynomial of virtual knots. I have included each of the authors' notations for their respective generalized polynomial invariants.

- Sawollek [**Saw**] (1999)

  $Z_D(x, y)$, the Conway polynomial and $\Delta_D(t)$, the (first) Alexander polynomial of a knot diagram D

- Silver and Williams [**SW1**] (2001)

  $\Delta_i(u_1, \ldots, u_d, v)$, the i-th virtual Alexander polynomial of a d-component virtual link

- Manturov [**Ma1**] (2001), [**Ma2**] (2004)

  the VA'-polynomial, which can be specialized to Δ, the Alexander polynomial

- Kauffman and Radford [**KR**], (2002)

  $G_K(s, t)$, the generalized Alexander polynomial for a knot K

- Fenn, Jordan-Santana, and Kauffman [**FJK**] (2004)

  Δ, the i-th Alexander polynomial

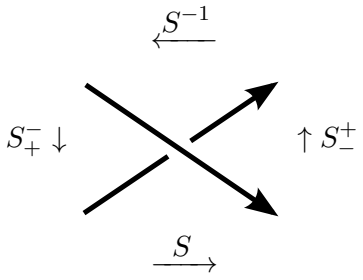
- Bartholomew and Fenn [**BF**], (2006)

  Δ_i, the i-th ideal polynomial, which can be specialized to the i-th Alexander polynomial

7.2 Comparison of Literature

Instead of proceeding chronologically, we will begin with the work of [**FJK**] and [**BF**]. The work done in the previous section is largely based on these papers and we will be able to make immediate connections between our work and the more general work contained in these articles. Moreover, it will be shown that essentially all other invariants listed above are specializations of [**FJK**]'s work.

- In [**FJK**], Fenn, Jordan-Santana, and Kauffman study what they call "switches", biracks, and biquandles. An abstract, set-theoretic definition of a switch is essentially any permutation on a set of elements which satisfies the Yang-Baxter equations. However, in practice, a switch is just a generalization of the operator which was called S in Section 6, which is thought of as acting on semi-arcs in the same way that as described in this paper. Keep in mind, though, that we assumed S was linear in Section 6, but there are interesting non-linear transformations one can consider in place of S. A birack is defined abstractly as as set with a switch which is invertible in each coordinate—in this sense, a birack is the most general algebraic object one can associate to a knot which will be invariant under Reidemeister moves of type II and III. For any birack, one can define "sideways" switches S^+_- and S^-_+, which are motivated by considering transformations between semi-arcs at a crossing in directions perpendicular to the direction which S acts. A diagram sums this up most appropriately:



  A biquandle is then defined as any birack for which the sideways switches preserve the diagonal (i.e., $S^+_-(x,x) = (y,y)$). This turns out to be precisely the property which corresponds to invariance under the first Reidemeister move (when S is thought to act on semi-arcs of a knot). Thus, a biquandle is the most general algebraic object one can associate to knots which is invariant under all Reidemeister moves. The authors give various examples of these objects, one of which is the *Alexander switch*, which we considered earlier without calling it

such:

$$S = \begin{pmatrix} 0 & B \\ C & 1 - BC \end{pmatrix}$$

This is used to define the *Alexander biquandle*, which is coincides exactly with the module constructed in Theorem 6.3.

- [**BF**] (Bartholomew and Fenn) focus on the specific case of linear switches. Section 6 is largely motivated by an attempt to flesh out some material in [**BF**], specifically the fact that the Alexander switch is the only linear switch with entries in a commutative ring. They prove an interesting result which we encountered earlier. Namely, that any linear switch which is a birack (i.e., satisfies axioms coming RII and RIII) is also a biquandle (i.e., is invariant under RI). They also develop the theory of linear switches over *non*-commutative rings by using the quaternions as an example. A particularly interesting result from their paper is Theorem 7.1 therein, which implies that any attempt to include a non-trivial relation at virtual crossings adds no new structure to a module constructed only by considering classical crossings. This is an important result when reading the work of [**Ma**] (also see [**CHN**], Remark 2).

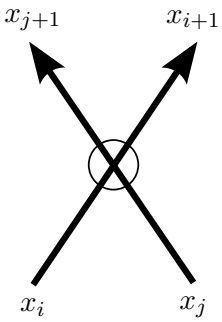
- Kauffman and Radford in [**KR**] develop the algebraic theory of biquandles without using the language of switches used by other authors. They do so more as a natural extension of the notion of quandles (see [**Bir**] for a quick reference on quandles). They nonetheless prove similar results to that in [**FJK**] about how the Alexander biquandle is the only linear biquandle and that the 0th generalized Alexander polynomial is a useful invariant for detecting non-classicality. They also introduce what they call bi-oriented quantum algebras, which provides an algebraic formulation of biquandles.

- Sawollek defines two polynomials in [**Saw**]. He calls them the Conway polynomial and the Alexander polynomial, respectively. However, it can be seen elsewhere (section 3 in [**SW2**] for example) that the Conway polynomial is essentially the same as the 0th generalized Alexander polynomial described above and that Sawollek's Alexander polynomial is just the first generalized Alexander polynomial. However, an interesting result from this paper is that

no normalization of the (first) Alexander polynomial satisfies any skein relation on virtual knot diagrams. This is in contrast to the classical case where the first Alexander polynomial can be equivalently defined via the Alexander-Conway polynomial, which is calculated via skein relations.

- In [**SW1**], Silver and Williams develop the ideas in their previous article [**SW0**] on Alexander groups. Their Alexander group is closely related to the Alexander module and is constructed directly from the and arcs crossings of a link diagram, much in same way as the method described at the end of Section 5 in this paper. Indeed, the abelianization of the Silver-Williams' Alexander group is exactly the Alexander module. They develop an "extended Alexander group" of virtual knots which, once abelianized, is equivalent to the generalized Alexander module. They use this theory to define a multivariable Alexander polynomial for virtual links, which is equivalent to the one described in Section 8 of this paper.

- Manturov's ([**Ma1**] and [**Ma2**]) appears as though he has defined a polynomial invariant unique from the others mentioned in this survey. He constructs a module from a knot by imposing the usual relations between arcs at classical crossings (i.e., $z = yt + (1-t)x$) and, in addition, he splits semi-arcs at virtual crossings and imposes the relation below.



$$x_{i+1} = x_i + \varepsilon$$
$$x_{j+1} = x_j - \varepsilon$$

All other invariants mentioned in this survey simply ignore virtual crossings. Thus, it appears as though Manturov's invariant might have some additional structure and be more powerful than other Alexander-like invariants. However, it a consequence of Theorem 7.1 in [**BF**] that

Manturov's module (constructed by imposing relations at virtual crossings) is equivalent to the same module without these extra relations.

8 A GENERALIZED MULTIVARIABLE ALEXANDER POLYNOMIAL FOR VIRTUAL LINKS

In this section we will define and (prove the invariance of) a generalized multivariable Alexander module (or biquandle) for virtual links. We build on the work done in Section 6 regarding the invariance of knot modules coming from linear switches. We follow the approach taken in the classical case by others by associating a different t variable to each component in a virtual link. This approach makes sense in the classical case because the abelianization of the link group for an l component link is $\mathbb{Z}^l$. The integral group ring of this group is exactly $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_l^{\pm 1}]$, and applying the Fox calculus yields a $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_l^{\pm 1}]$-module from which the multivariable Alexander polynomials can be derived. This construction carries over rather straightforwardly to the virtual theory.

The invariants we construct in this section are equivalent to those considered by [**SW1**]. However, Silver and Williams construction is not done general context of switches/biquandles developed by [**FJK**]. A remark by [**KR**] on p. 23 of their paper mentions the construction of a multivariable biquandle invariant, though no further inquiry is made. Furthermore, in section 5 of their paper, [**BF**] make the remark "Assume for simplicity that we are dealing with a knot. The link case is similar and details can safely be left to the reader." The purpose of this section is to make the connection between Silver and Williams' approach and more explicitly study the invariants alluded to, but not developed, by other authors.

8.1 The Generalized Multivariable Alexander Module

Definition: Let L be a virtual link with l components and n classical crossings. Label the semi-arcs of L by $x_1, \ldots, x_{2n}$ and assign to each component a unique t-variable, indexed $t_1, \ldots, t_l$. The *generalized multivariable Alexander module* of L (or the *multivariable Alexander biquandle* in the language of [**FJK**]), denoted $\mathfrak{M}_L$, is the $\mathbb{Z}[s^{\pm 1}, t_1^{\pm 1}, \ldots, t_l^{\pm 1}]$-module generated by $x_1, \ldots, x_{2n}$ modulo the following relations, induced by each classical crossing of L:

$$z = sy$$
$$w = t_o x + (1 - st_u)y$$

where x, y, z, and w are appropriately labeled elements from $\{x_1, \ldots x_{2n}\}$ and t_o is the t-variable associated to the *overstrand* and t_u is the t-variable associated to the *understrand* in each crossing.

Using the language of the authors who have written on this subject, we could equivalently describe this invariant in the following ways:

Silver and Williams:	the abelianization of the extended Alexander group
[**FJK**], [**KR**]:	multivariable Alexander biquandle
[**BF**], current paper:	(generalized) multivariable Alexander module

Theorem 9.1. $\mathfrak{M}_L$ is an invariant of virtual links.

Proof: The virtual and semi-virtual Reidemeister moves do not affect the labeling of semi-arcs or the relations of $\mathfrak{M}_L$. Thus the presentation of $\mathfrak{M}_L$ is only changed by classical Reidemeister moves.

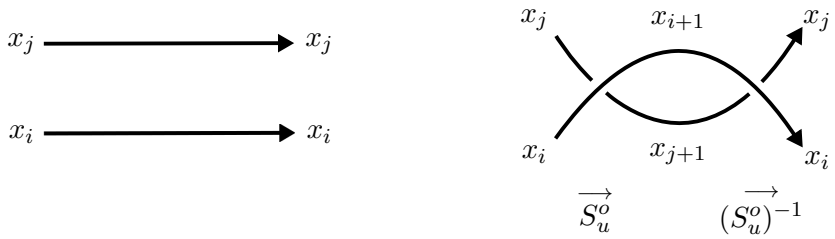
Let S_u^o be the *multivariable Alexander switch*, defined by

$$S_u^o = \begin{pmatrix} 0 & s \\ t_o & 1 - st_u \end{pmatrix}$$

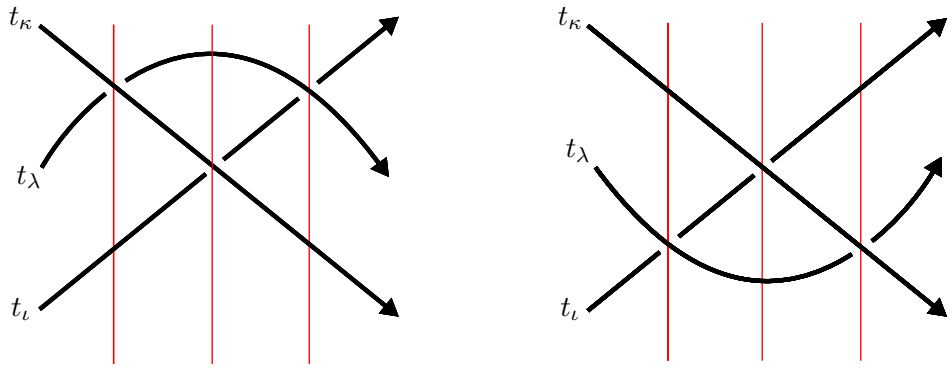
For positive crossings, we can think of the the left-to-right action across crossings on the semi-arcs as S_u^o, and the right-to-left action as $(S_u^o)^{-1}$. For negative crossings, the left-to-right action is $(S_u^o)^{-1}$ and the right-to-left action is S_u^o.

Since only one component of L is involved in a type I Reidemeister move, proving the invariance of $\mathfrak{M}_L$ under RI is precisely the same as in the module which was constructed in Section 6.2.

Similarly, since the same two components are involved at each crossing in a type II Reidemeister move, the proof regarding RII in Section 7.2 applies here as well.



The case of RIII is unique in the multivariable case. We must assume that three arbitrary components of the link L are involved in the type III Reidemeister move. In the diagram below, the labels t_ι, t_λ, and t_κ are the labels of the *components*, not the semi-arcs, where $\iota, \lambda, \kappa \in \{1, \ldots, l\}$.



$(id \times S^\kappa_\lambda)(S^\kappa_\iota \times id)(id \times S^\lambda_\iota)$ $\qquad$ $(S^\lambda_\iota \times id)(id \times S^\kappa_\iota)(S^\kappa_\lambda \times id)$

The action on the semi-arcs across as they traverse crossings is indicated by the operators listed below each figure. It is certainly not immediate that the outgoing semi-arcs in each figure have the same labels.

One must pay particularly close attention to the fact that, when we write the operators in their matrix form, we cannot simply concatenate them in the order we see above. We must multiply the matrices in the order in which the operators are composed. Thus, in matrix form, the transformation on the labels between incoming and outgoing semi-arcs in the first diagram is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & s \\ 0 & t_\lambda & 1-st_\iota \end{pmatrix} \begin{pmatrix} 0 & s & 0 \\ t_\kappa & 1-st_\iota & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & s \\ 0 & t_\kappa & 1-st_\lambda \end{pmatrix}$$

For the second diagram, the transformation is

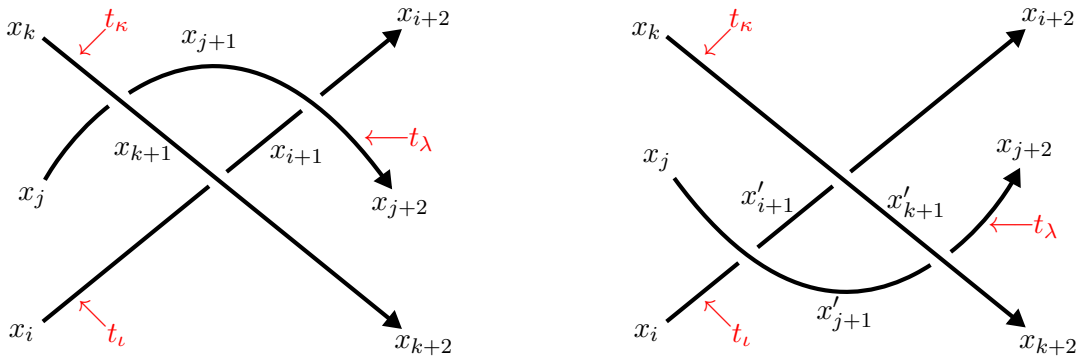
$$\begin{pmatrix} 0 & s & 0 \\ t_\kappa & 1-st_\lambda & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & s \\ 0 & t_\kappa & 1-st_\iota \end{pmatrix} \begin{pmatrix} 0 & s & 0 \\ t_\lambda & 1-st_\iota & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Fortunately, both of these products are equivalent to

$$\begin{pmatrix} 0 & 0 & s^2 \\ 0 & st_\kappa & s(1-st_\lambda) \\ t_\kappa t_\lambda & t_\kappa(1-st_\iota) & 1-st_\iota \end{pmatrix}$$

(See [**Wolf**] for calculation.) Thus, the labels of the outgoing semi-arcs in each diagram coincide.

We still need to check that the differences between the labeling on the inner arcs (x_i, x_j, x_k and x'_i, x'_j, x'_k in the diagram below) does not change $\mathfrak{M}_L$.



Much in the same way that we proved the invariance of the module in Section 7, the invariance of $\mathfrak{M}_L$ reduces to demonstrating that the following matrices are equivalent:

$$\begin{pmatrix} t_\kappa & 0 & s(1-st_\iota) \\ 0 & t_\kappa & 1-st_\lambda \\ 0 & 0 & s \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} t_\lambda & 1-st_\iota & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{pmatrix}$$

However, this is almost immediately obvious. One can add appropriate multiples of rows together to reduce the first matrix to $\mathrm{diag}\{t_\kappa, t_\kappa, s\}$ and the second matrix to $\mathrm{diag}\{t_\lambda, s, s\}$. After multiplying by suitable inverses, both can be reduced to the identity matrix.

■

8.2 The Generalized Multivariable Alexander Polynomials

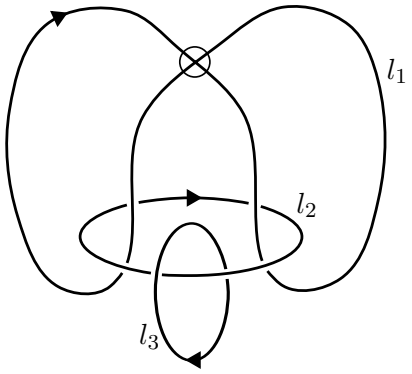
Definition: The i-th elementary ideals of $\mathfrak{M}_L$ are generated by the i-th minors of M. We can define the *i-th generalized multivariable Alexander polynomial* as an element $\Delta_i(s, t_1, \ldots, t_l)$ which generates the i-the elementary ideal of $\mathfrak{M}_L$. We say *an* element since $\langle \Delta_i(s, t_1, \ldots, t_l) \rangle = \langle s^{m_0} t_1^{m_1} \cdots t_l^{m_l} \Delta_i(s, t_1, \ldots, t_l) \rangle$ for any $m_0, m_1, \ldots, m_l \in \mathbb{Z}$.

Remarks

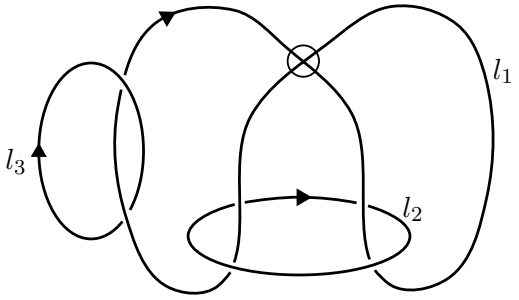
- Note that setting $t_1 = \cdots = t_l = t$ returns the generalized Alexander polynomial.
- Setting $s = 1$ returns the multivariable Alexander polynomial
- Setting both $t_1 = \cdots = t_l = t$ and $s = 1$ returns the Alexander polynomial.

8.3 Example

This example has been adapted from Example 6.4 in [**SW2**]. Consider the following two virtual links, L_1 and L_2. The links intuitively appear to be distinct, however no invariants introduced before this section are sufficient to prove this.



L_1



L_2

Assign the variables t, u, v to the components l_1, l_2, l_3 (respectively) in each diagram. We will denote regular i-th multivariable Alexander polynomial by $\Delta_i(t, u, v)$, and the i-th generalized multivariable Alexander polynomial by $\Delta_i(s, t, u, v)$. The values for $i = 0$ and 1 of these invariants

for the links above is expressed in the table below (the polynomials are trivial for both links for $i > 1$).[14]

	$\Delta_0(t,u,v)$	$\Delta_1(t,u,v)$	$\Delta_0(s,t,u,v)$	$\Delta_1(s,t,u,v)$
L_1	0	$u-1$	$(s-1)(t-1)(st-1)(su-1)^2$	$su-1$
L_2	0	$t-1$	$(s-1)(t-1)(st-1)^2(su-1)$	$st-1$

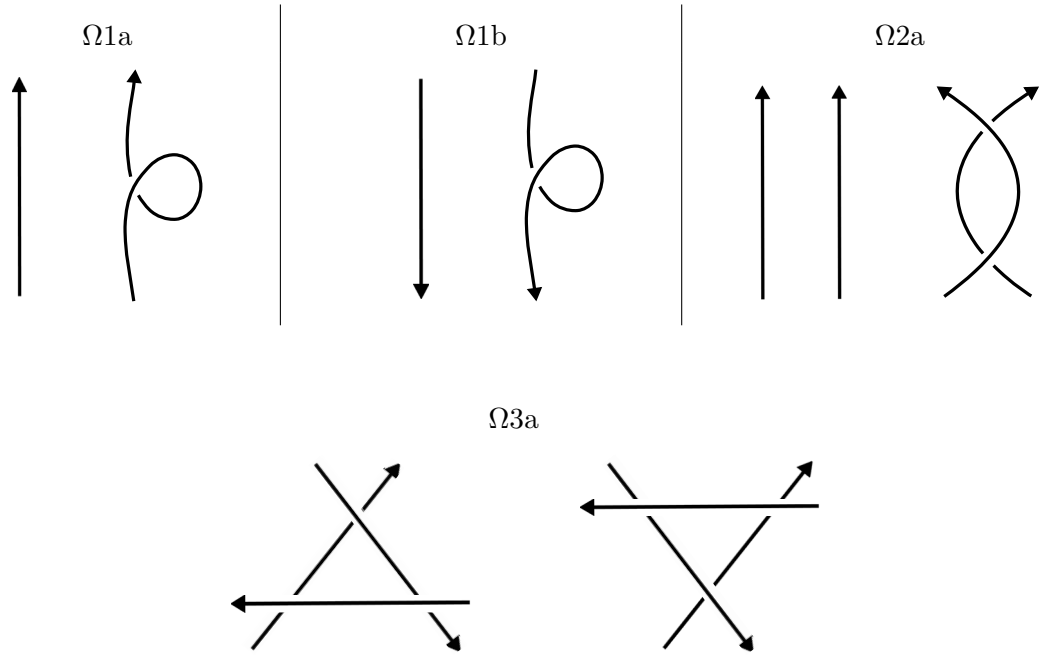
Note that in both the regular and generalized case, the first Alexander polynomials are distinct, however not distinct enough to conclude whether the links differ by more than a simple relabeling of components. Moreover, note that the (single t-variable) generalized Alexander polynomials (which can be found by setting $t = u = v$) do not distinguish the two links either. However, the value of $\Delta_0(s,t,u,v)$ guarantees the two links are indeed distinct (since no permutation of t, u, v make the polynomials equal). Thus, only with the full power of the generalized multivariable Alexander polynomial can these links be distinguished.

[14]The calculation of these values can be found as a Mathematica Notebook or PDF at [Mil]

9 POLYAK'S GENERATING SETS OF ORIENTED REIDEMEISTER MOVES

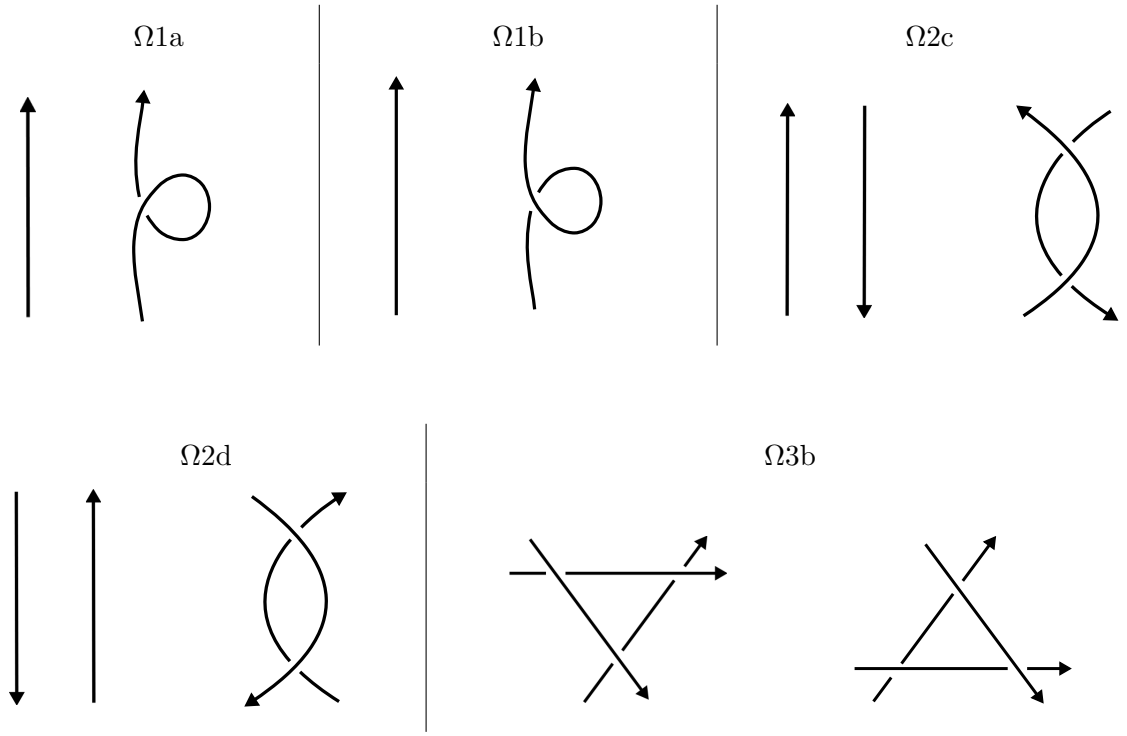
As mentioned earlier, there are a total of 16 possible oriented Reidemeister moves (4 type I moves, 4 type II moves, and 8 type III moves). Polyak [**Pol**] has found a set of four of these moves which generate the other 12. This set is shown below (called Set A), along with Polyak's naming system for each of the moves.

Set A



Another generating set of five Reidemeister moves found by Polyak is below.

Set B



Polyak demonstrates that each of the sets above are minimal, in the sense that any proper subset of either does not generate all other Reidemeister moves.

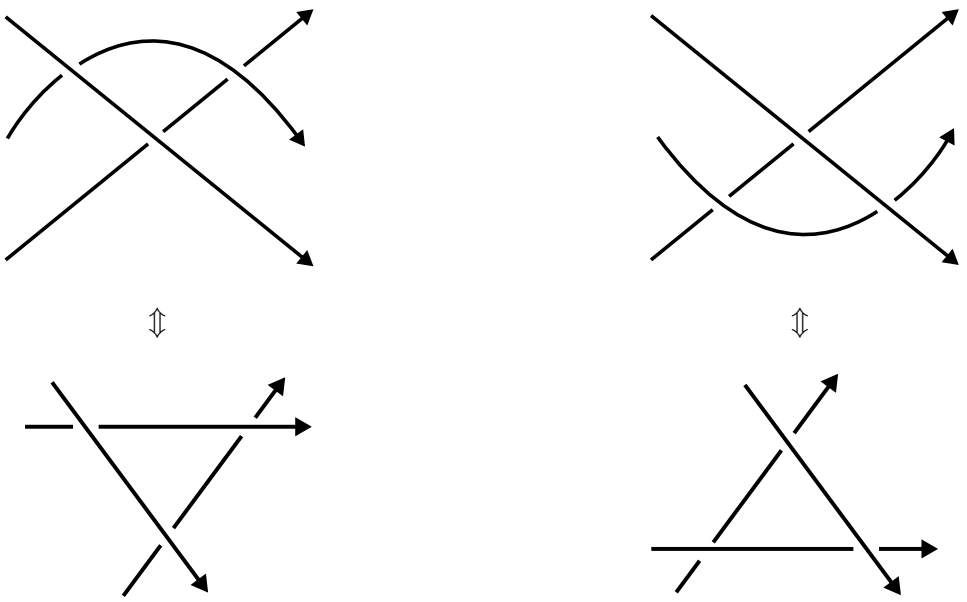
It turns out that the moves which were considered in Section 6 do not generate all possible Reidemeister moves. We will use this section to address this error in the arguments in that section. Other authors attempt to address this issue in their paper as well. [**FJK**], for example, make the effort point out that type II Reidemeister moves can be used on type III moves to ensure all strands point in the same direction. However, in their paper, they only address this kind of adjustment for moves of the form Ω3a, whereas their arguments regarding the type III Reidemeister move utilized the Ω3b move. It is not immediately obvious whether enough has been shown by [**FJK**] to show invariance under all possible orientations. A similar issue exists in [**BF**]. Polyak's work could be used to iron out these arguments in an efficient manner. Specifically, one must only show that an invariant holds for the 4 diagrams in Set A or the 5 diagrams in Set B. We will apply this principle in the present paper, showing that the module considered in Section 6 is invariant under the moves in Set B. This discussion was delayed until now because we must introduce the notion of *sideways operators*, which would have over-encumbered the narrative in Section 6.

9.1 Module Invariance (Revisited)

Completion of proof of Theorem 6.3:

Let $\mathcal{M}$ be the module associated to any virtual knot by the process described in Theorem 6.3. Note that in discussion leading up to the statement of that theorem, we already considered Polyak's moves Ω1a and Ω3b.

Equivalence of move considered in Section 6 and Polyak's Ω3b

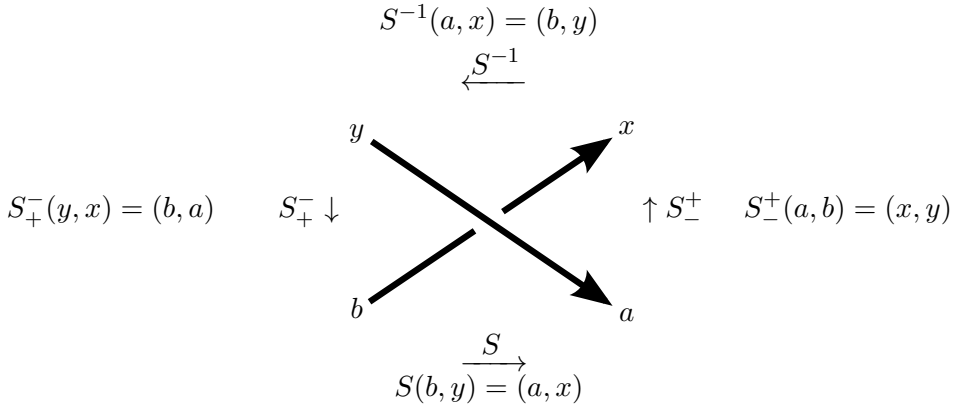


Thus, it remains to prove invariance under Ω1b, Ω2c, and Ω2d. Invariance under Ω1b can be used using the same arguments in Section 6, simply by traversing the crossing in the reverse direction (since the crossing is negative).

Sideways Operators

To address the type II moves, we revisit the notion of *sideways operators* (mentioned in Section 7.2). For the general definition of the sideways operator of a linear switch see [**BF**]; for the definition of sideways operator for an arbitrary switch see [**FJK**]. Sideways operators are simply the relationship between the pairs of arcs (generators) at a crossing orthogonal to those acted on by a switch. For example, if a switch S has a left-to-right action on arcs at positive crossings, the sideways operator,

denoted S^+_-, has bottom-to-top action. Similarly, we can denote the transformation for the top-to-bottom action by S^-_+. The diagram below demonstrates the relations between pairs of arcs across any direction at a positive crossing.



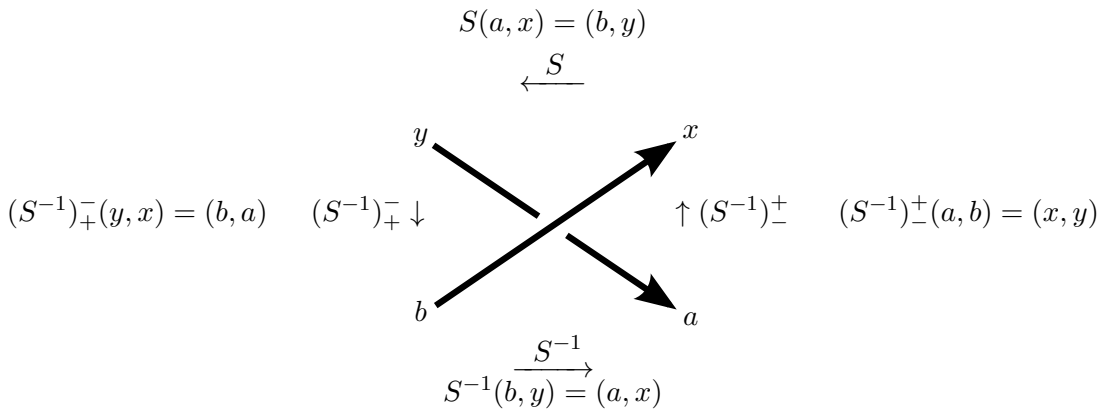
Note that, as defined, S^+_- and S^-_+ are not exact inverse operators. This is due to the order of the ordered pairs (y,x) and (x,y). However, we can easily find $(S^+_-)^{-1}$ in terms of S^-_+. If we let T be the twist operator defined by $T(w,z) \mapsto (z,w)$, then we have

$$(S^+_-)^{-1}(x,y) = (a,b) = T(b,a) = T(S^-_+(y,x)) = T(S^-_+(T(x,y)))$$

Similarly, we can see

$$(S^-_+)^{-1}(b,a) = (y,x) = T(x,y) = T(S^+_-(a,b)) = T(S^+_-(T(b,a)))$$

Remembering that, at negative crossings, the roles of S and S^{-1} are interchanged, we can also define sideways maps $(S^{-1})^+_-$ and $(S^{-1})^-_+$ by the diagram below.



There is an important relationship between the sideways matrices of positive and negative crossings. Namely, that $(S^+_-)^{-1} = (S^-_+)^{-1}$ and $(S^-_+)^{-1} = (S^+_-)^{-1}$. This can be seen by rotating the crossing of the negative diagram 180 degrees and relabeling semi-arcs to correspond with the positive crossing.

Sideways Operators of the Alexander Switch

Note that for the switch considered in Theorem 6.3 (called the Alexander switch):

$$S = \begin{pmatrix} 0 & B \\ C & 1 - BC \end{pmatrix}$$

we can find explicit formulas for S^+_- and S^-_+. To do this, suppose that we know the labels of the arcs a and b in the positive crossing diagram above and want to solve for the labels of x and y. We know the relation between these arcs will be $S(b,y) = (a,x)$, or

$$By = a$$
$$Cb + (1 - BC)y = x$$

from which we can see $y = B^{-1}a$ and thus $x = (B^{-1} - C)a + Cb$. This can be expressed as

$$\begin{pmatrix} B^{-1} - C & C \\ B^{-1} & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Note this is exactly the relationship between the arcs acted on by the sideways operation S^+_-. Thus

we have found

$$S_-^+ = \begin{pmatrix} B^{-1} - C & C \\ B^{-1} & 0 \end{pmatrix}$$

S_+^- can be calculated in a similar manner to obtain

$$S_+^- = \begin{pmatrix} B - C^{-1} & C^{-1} \\ B & 0 \end{pmatrix}$$

Note that, in matrix form, the twist operator T is given by

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Hence, the inverse sideways matrices are given by

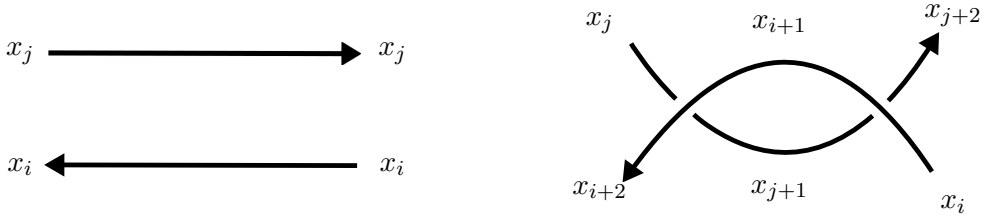
$$(S^{-1})_-^+ = (S_+^-)^{-1} = T(S_-^+)T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} B^{-1} - C & C \\ B^{-1} & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & B^{-1} \\ C & B^{-1} - C \end{pmatrix}$$

and

$$(S^{-1})_+^- = (S_-^+)^{-1} = T(S_+^-)T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} B - C^{-1} & C^{-1} \\ B & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & B \\ C^{-1} & B - C^{-1} \end{pmatrix}$$

Invariance of $\mathcal{M}$ Under Ω2c and Ω2d

We are now in a position to consider the effect of Reidemeister moves Ω2c and Ω2d on the module $\mathcal{M}$. Consider the following diagram of Ω2c, with semi-arcs labeled.

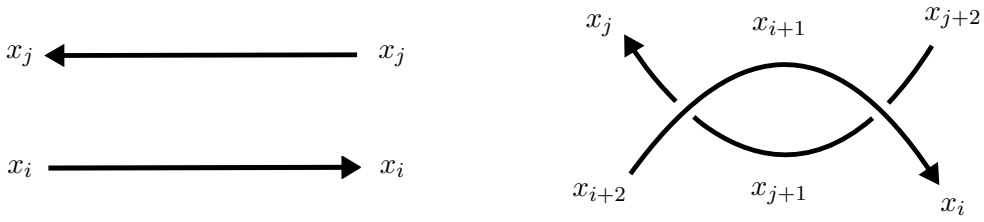


We must check that the semi-arcs $x_{i+2} = x_i$ and $x_{j+2} = x_j$. Note that, using the notation of sideways matrices, we can write the relations between arcs traveling from left-to-right in the

diagram above as $S^+_-(S^{-1})^-_+$. Recalling the identities we just established about sideways matrices, we obtain

$$S^+_-(S^{-1})^-_+ = S^+_-(S^+_-)^{-1} = id$$

This ensures that the arcs on the left and right side of the diagram must coincide in each diagram. Similarly, for the case of Ω2d we have the diagram



The left-to-right transformation on semi-arcs is then given by $S^-_+(S^{-1})^+_-$. Again this reduces to the identity:

$$S^-_+(S^{-1})^+_- = S^-_+(S^-_+)^{-1} = id$$

As for the fact that the two diagrams present isomorphic modules, note that in each case, the module presented by the figure on the right has the form

$$\begin{pmatrix} & M & & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & -1 & 0 & Y \\ \mathbf{0} & -1 & 0 & X & Z \end{pmatrix}$$

where M the module presented by the figures on the left and $X = B^{-1}, Y = C, Z = B^{-1} - C$ in the case of Ω2c and $X = B, Y = C^{-1}, Z = B - C^{-1}$ in the case of Ω2d. In both cases, X and Y are units, and hence differ from M by two applications of the **T4** transformation.

This completes the proof that $\mathcal{M}$ is invariant under Polyak's generating set of Reidemeister moves, Set B. Thus, $\mathcal{M}$ is an invariant of oriented virtual knots.

■

Corollaries

- The generalized Alexander polynomials are invariants of oriented virtual knots.

- This proof can be easily modified to fully establish the invariance of the generalized *multi-variable* Alexander module (and derived polynomials) introduced in Section 9. This makes up for the subtlety about arc orientations not addressed in that section.

10 QUESTIONS FOR FURTHER RESEARCH

Polynomial Invariants of Long Virtual Knots

There are also several other papers which study Alexander-like invariants of *long* virtual knots. The benefit of studying long virtual knots is (1) that their connected sum is well defined (whereas the connected sum between regular virtual knots depends on the point at which each knot is cut) and (2) the Alexander polynomial can be normalized. We will not discuss this topic at length here, but the literature regarding these invariants should be mentioned. Here are some authors writing on this subject with references to their papers.

- Silver and Williams [**SW3**] (2004)
- Jana Archibald [**Ar**] (2009)
- Afanasiev [**Af**] (2009)
- Brandenbursky [**Br**] (2010)
- Crans Henrich Nelson [**CHN**] (2011)

In her dissertation, Jana Archibald establishes some interesting properties of her *virtual multivariable Alexander polynomial* (vMVA). Specifically, she demonstrates that the coefficients of her vMVA are finite type (Vasilliev) invariants. Furthermore , her vMVA satisfies some local relations which are similar to those which the normalized classical Alexander polynomial satisfies. The development of a normalized virtual Alexander polynomial (i.e., not only defined up to multiplication by units) is of particular interest in the study of the Alexander polynomial's relationship with other invariants. It is unknown exactly how Archibald's research applies to the work done in this paper. It is also unknown what the relationship between the various long virtual knot invariants developed by the authors above relate. This would be an interesting avenue of research and one which is worth exploring.

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